Fall 2015

Data Structures Lecture 15

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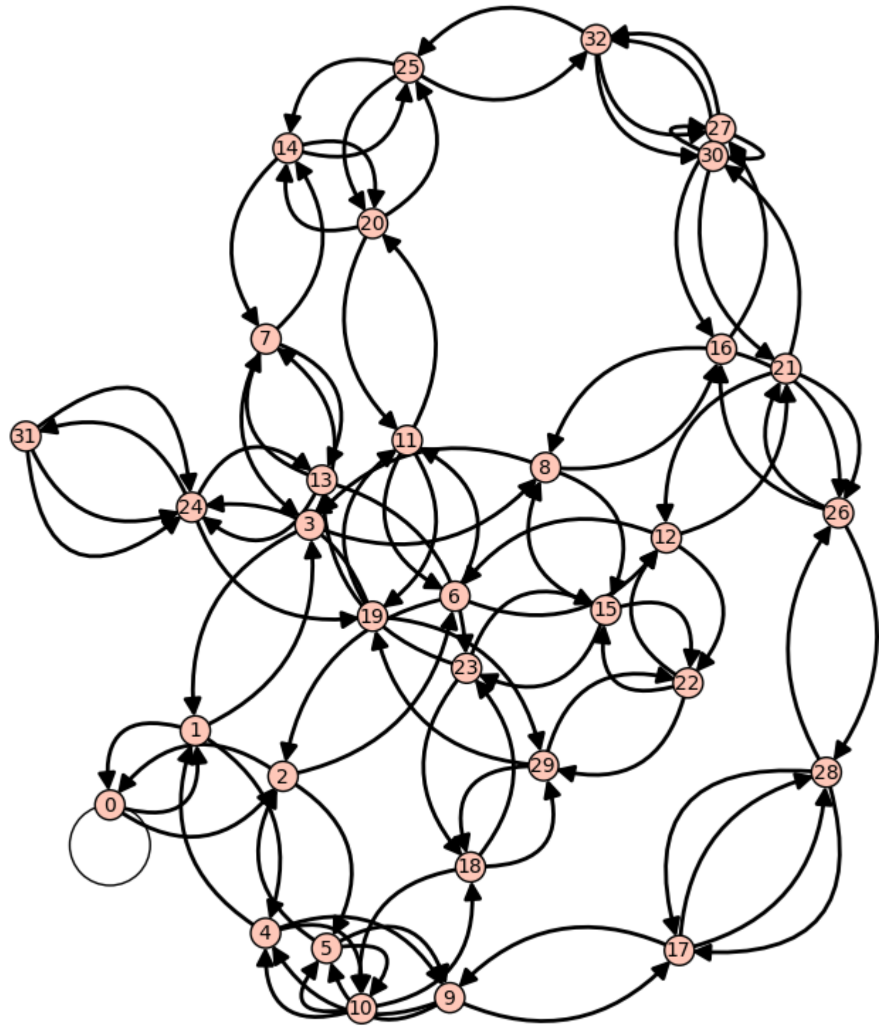
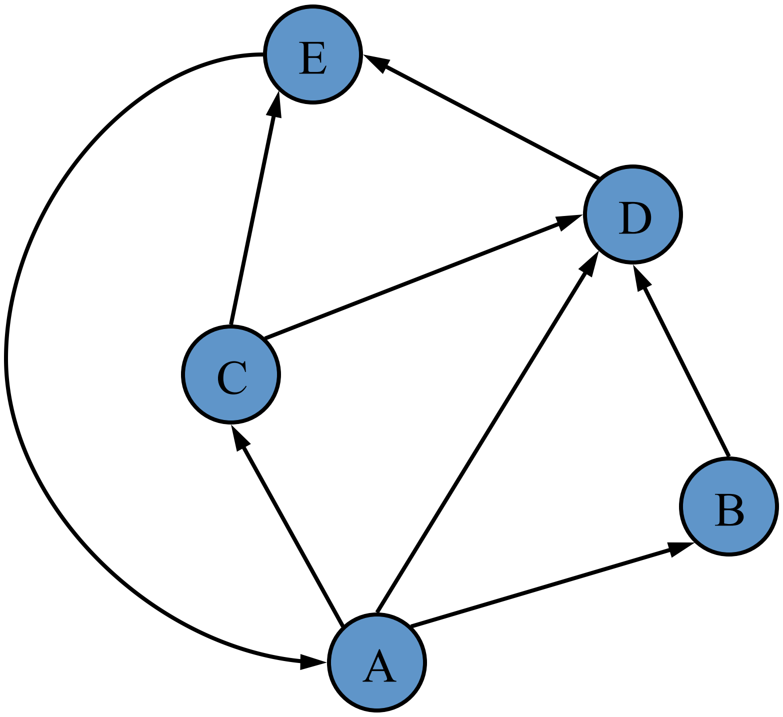


Graphs II

Digraphs, Strongly Connective Component, Topological Sorting, and Minimum Spanning Tree

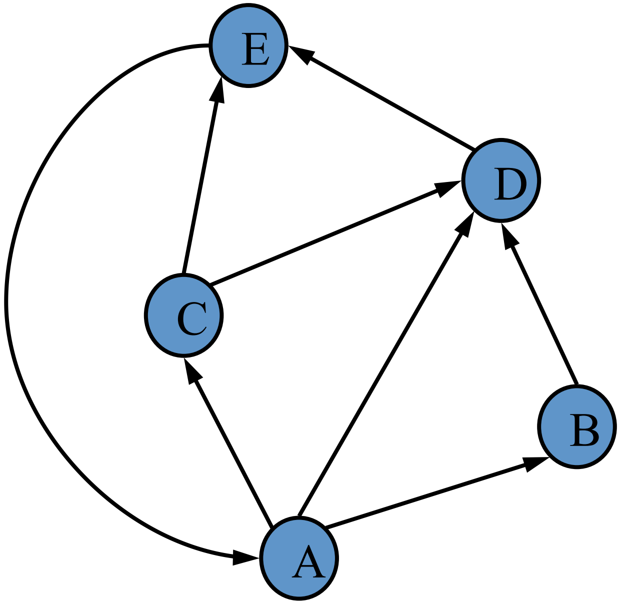
Digraphs

- A digraph is a graph whose edges are all directed
  - Short for “directed graph”
- Applications
  - one-way streets
  - flights
  - task scheduling



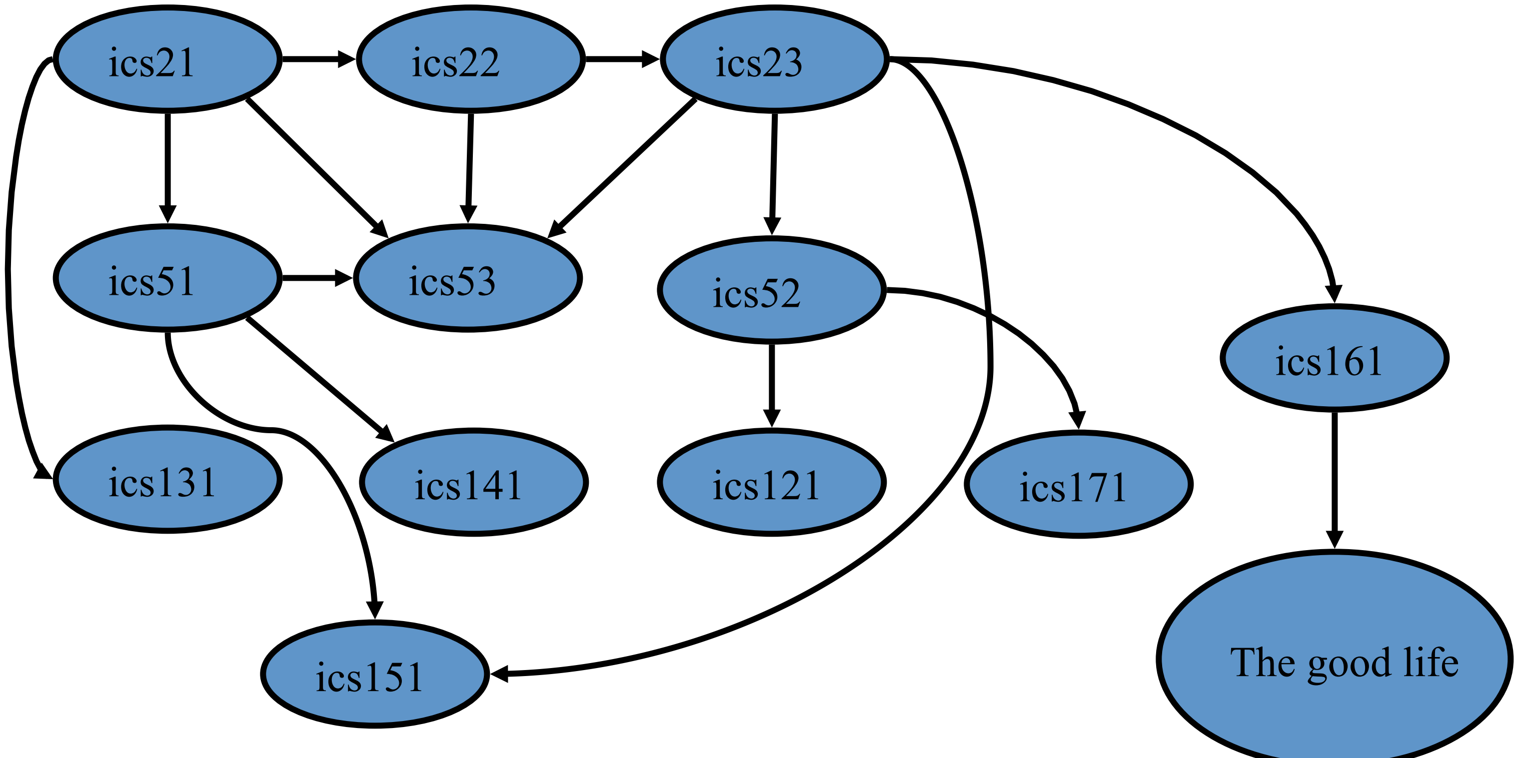
Digraph Properties

- A graph G=(V,E) such that
  - Each edge goes in one direction:
  - Edge (a,b) goes from a to b, but not b to a
- If G is simple, $m \leq n \cdot (n - 1)$
- If we keep in-edges and out-edges in separate adjacency lists, we can perform listing of incoming edges and outgoing edges in time proportional to their size



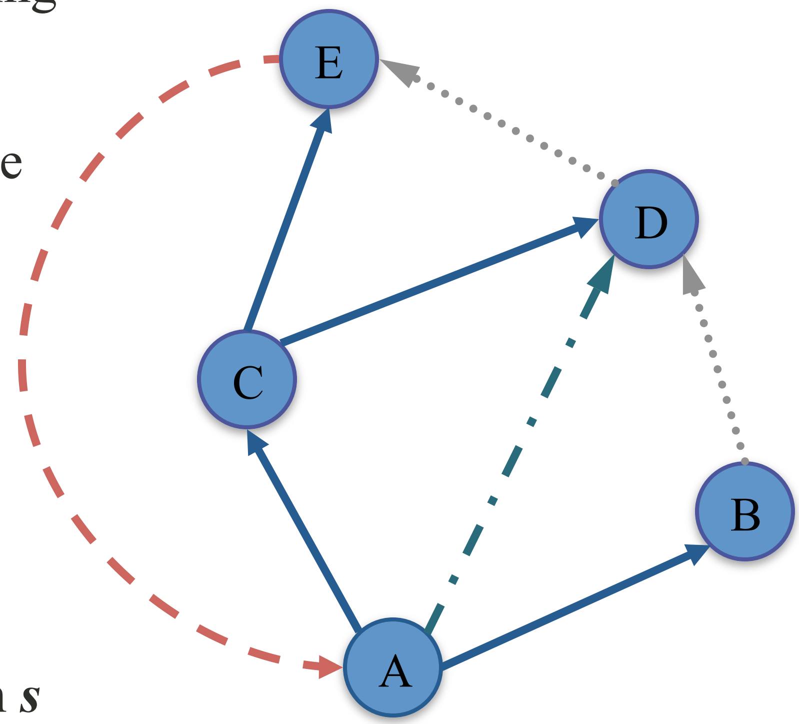
Digraph Application

- Scheduling: edge (a,b) means task a must be completed before b can be started



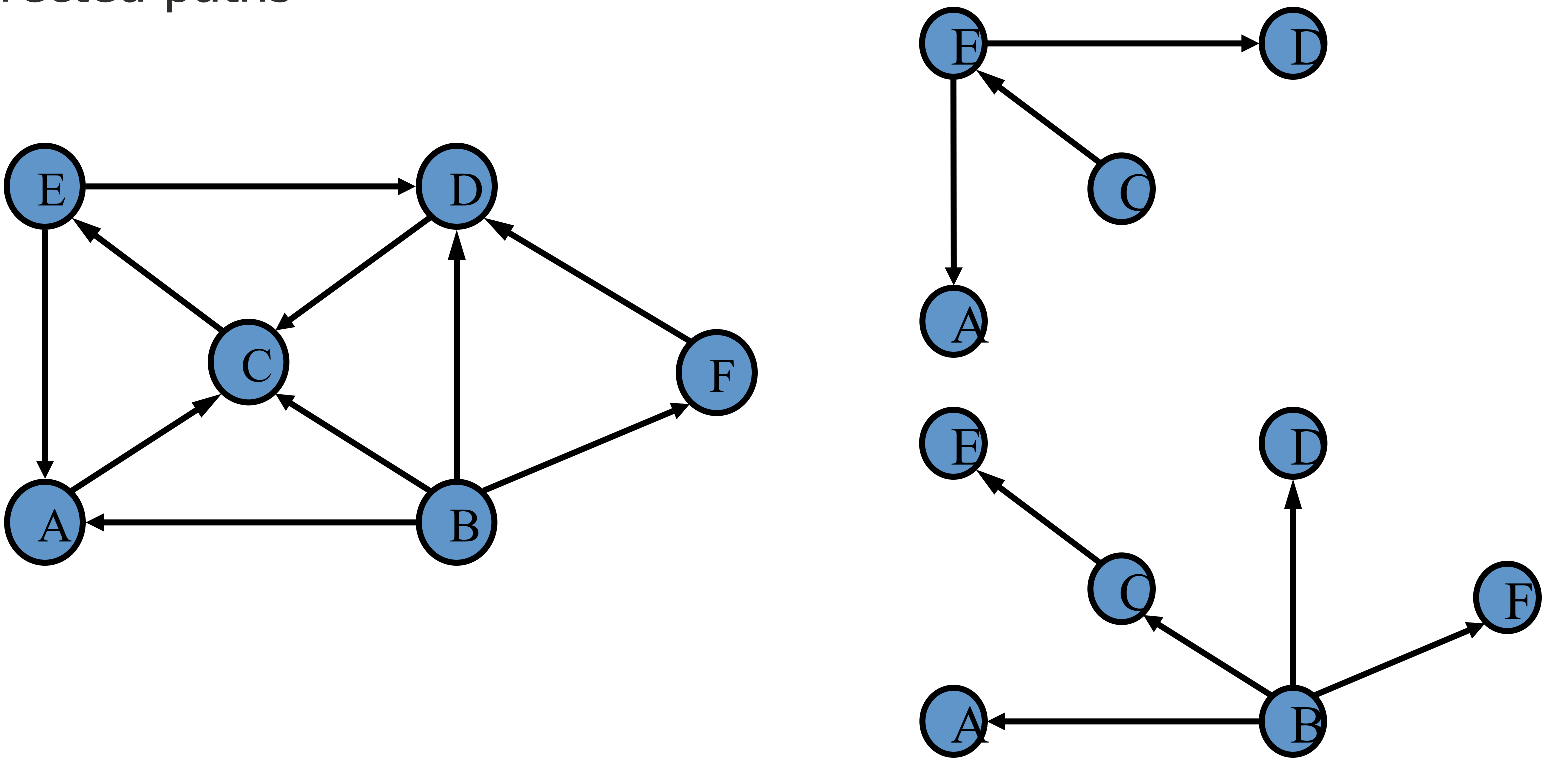
Directed DFS

- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction
- In the directed DFS algorithm, we have four types of edges
  - discovery edges
  - back edges
  - forward edges
  - cross edges
- A directed DFS starting at a vertex ***s*** determines the vertices reachable from ***s***



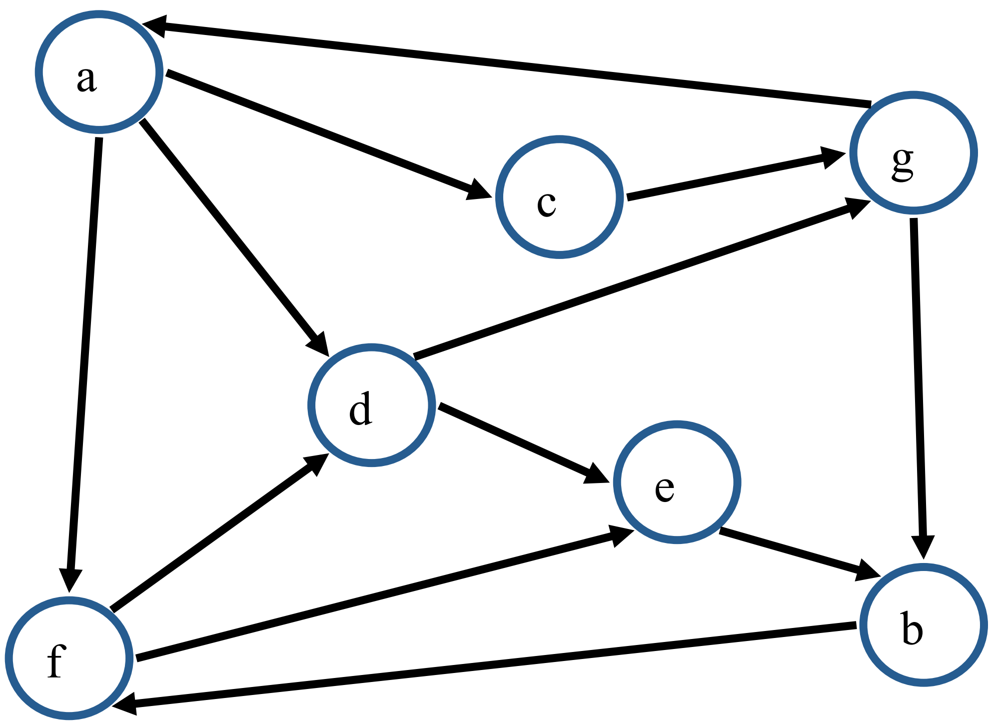
Reachability

- DFS tree rooted at v: vertices reachable from v via directed paths



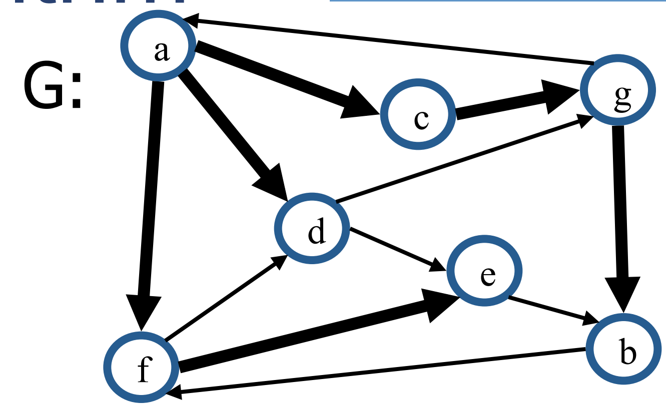
Strong Connectivity

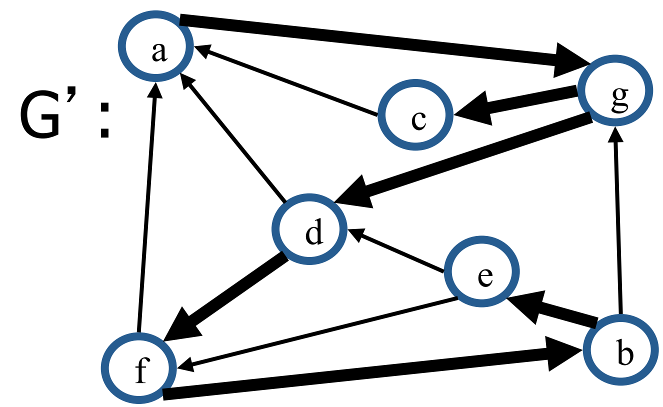
- Each vertex can reach all other vertices



Strong Connectivity Algorithm

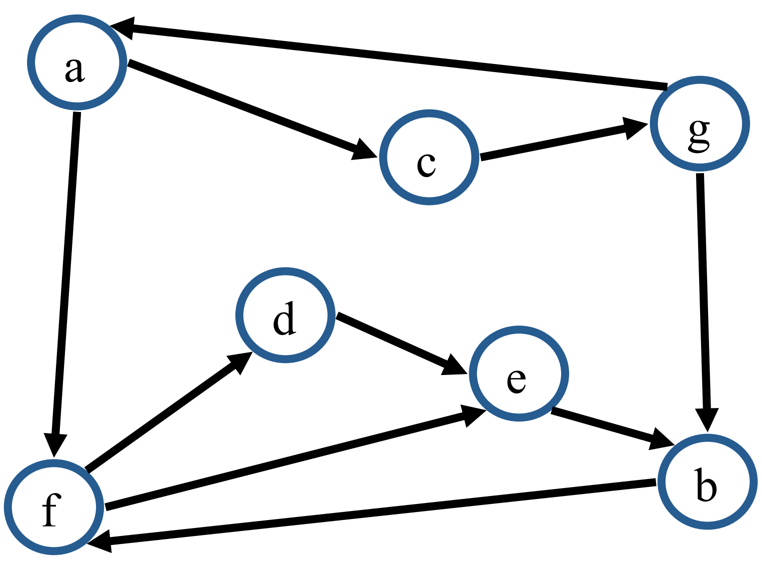
- Pick a vertex v in G
- Perform a DFS from v in G
  - If there's a w not visited, print "no"
- Let G' be G with edges reversed
- Perform a DFS from v in G'
  - If there's a w not visited, print "no"
  - Else, print "yes"
- Running time: O(n+m)





Strongly Connected Components

- Maximal subgraphs such that each vertex can reach all other vertices in the subgraph
- Can also be done in O(n+m) time using DFS, but is more complicated (similar to biconnectivity).

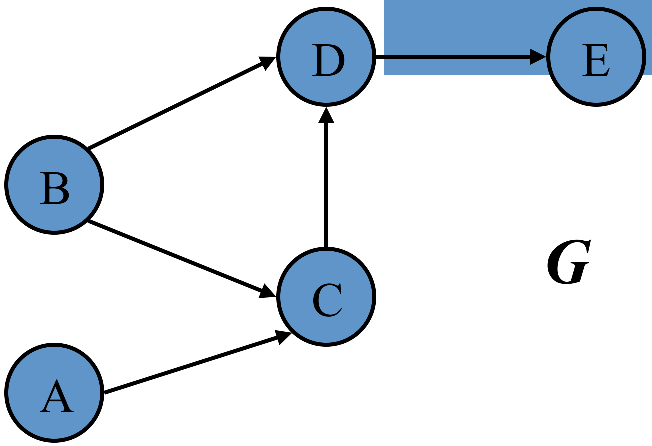


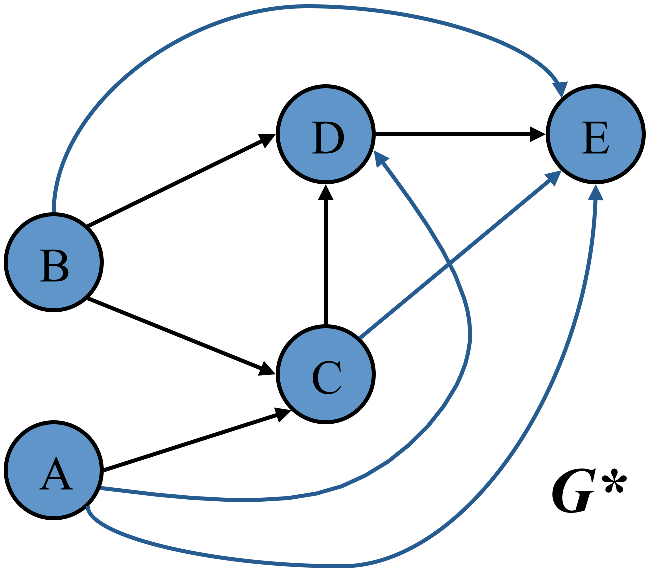
{ a , c , g }

{ f , d , e , b }

Transitive Closure

- Given a digraph G, the transitive closure of G is the digraph G^* such that
  - G^* has the same vertices as G
  - if G has a directed path from u to v ($u \neq v$), G^* has a directed edge from u to v
- The transitive closure provides reachability information about a digraph

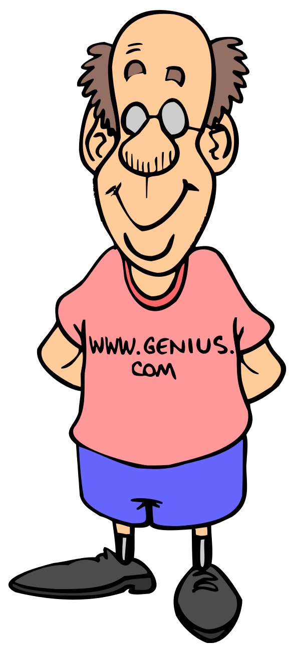




Computing the Transitive Closure

- We can perform DFS starting at each vertex
  - O(n(n+m))

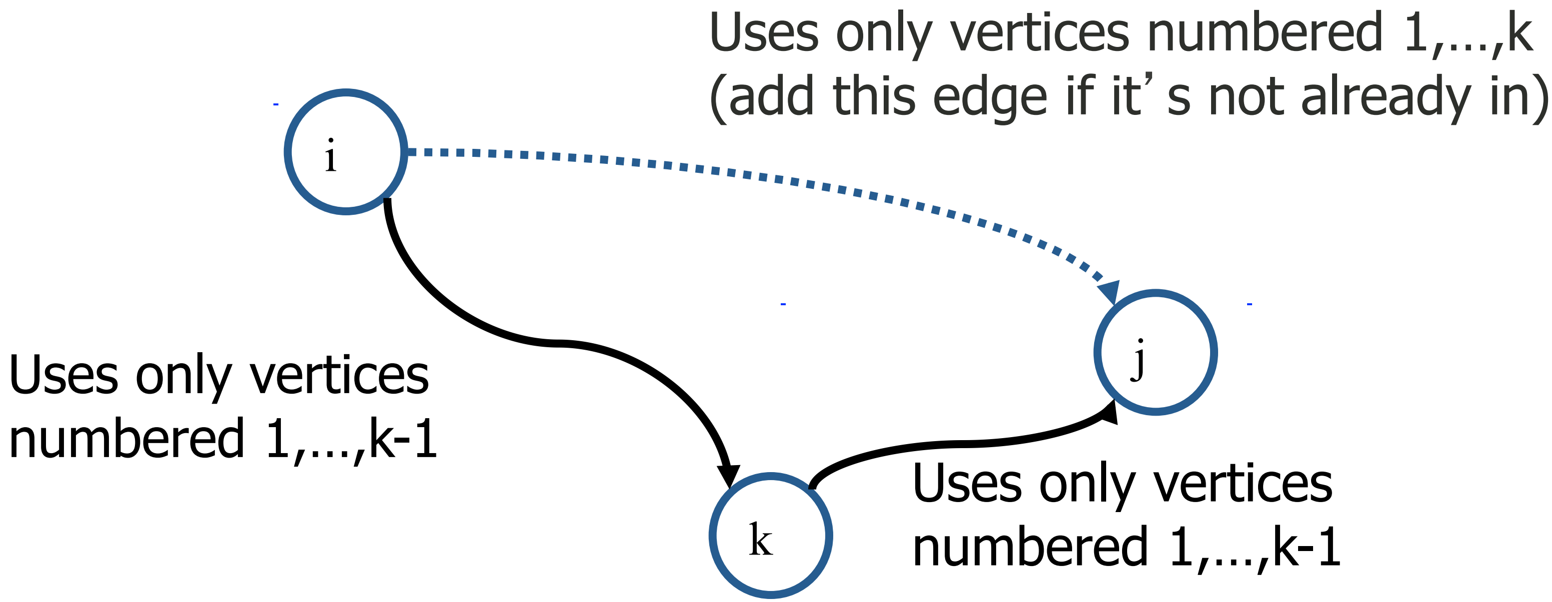
If there's a way to get from A to B and from B to C, then there's a way to get from A to C.



Alternatively ... Use dynamic programming: The Floyd-Warshall Algorithm

Floyd-Warshall Transitive Closure

- Idea #1: Number the vertices 1, 2, …, n.
- Idea #2: Consider paths that use only vertices numbered 1, 2, …, k, as intermediate vertices:

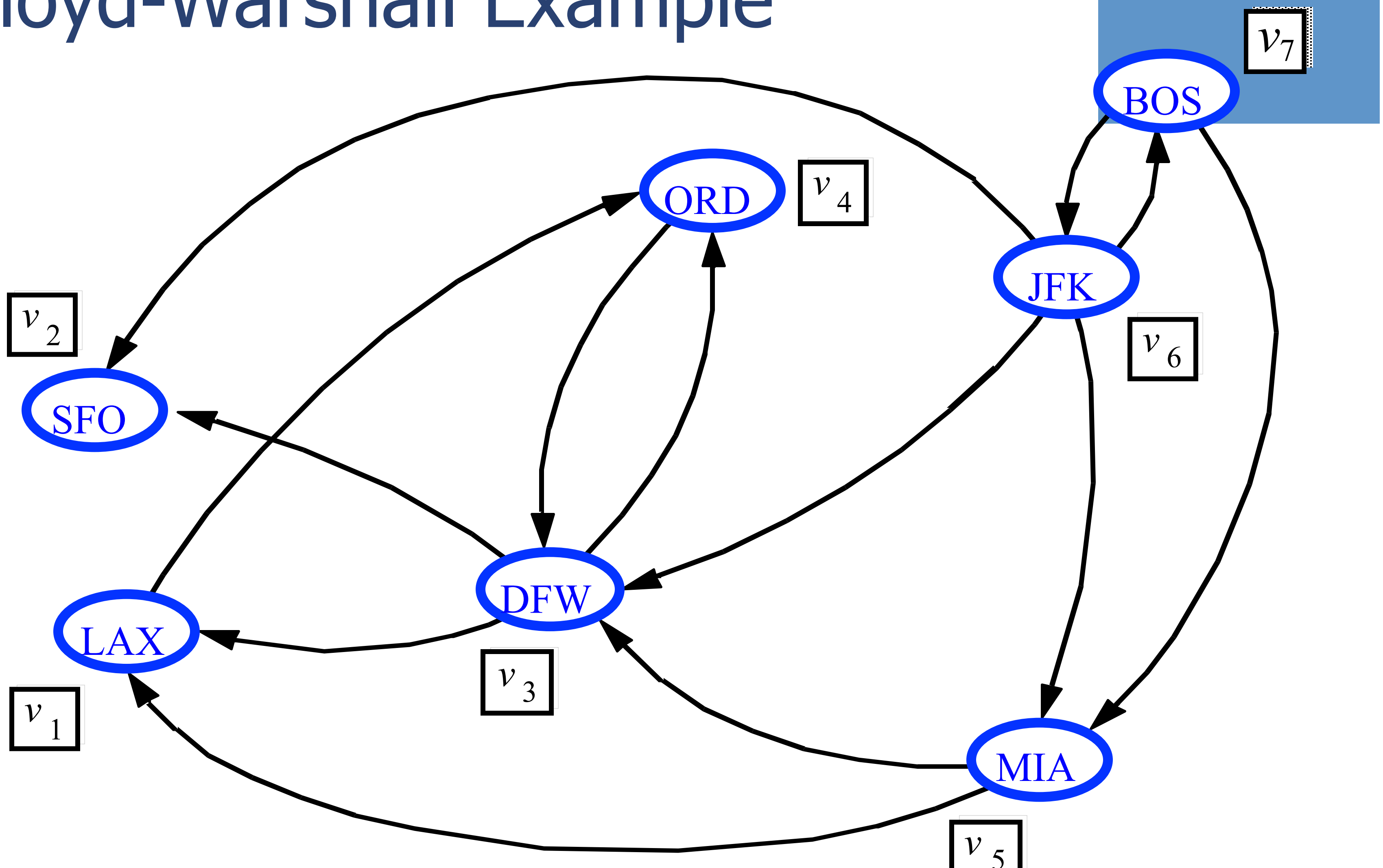


Floyd-Warshall's Algorithm

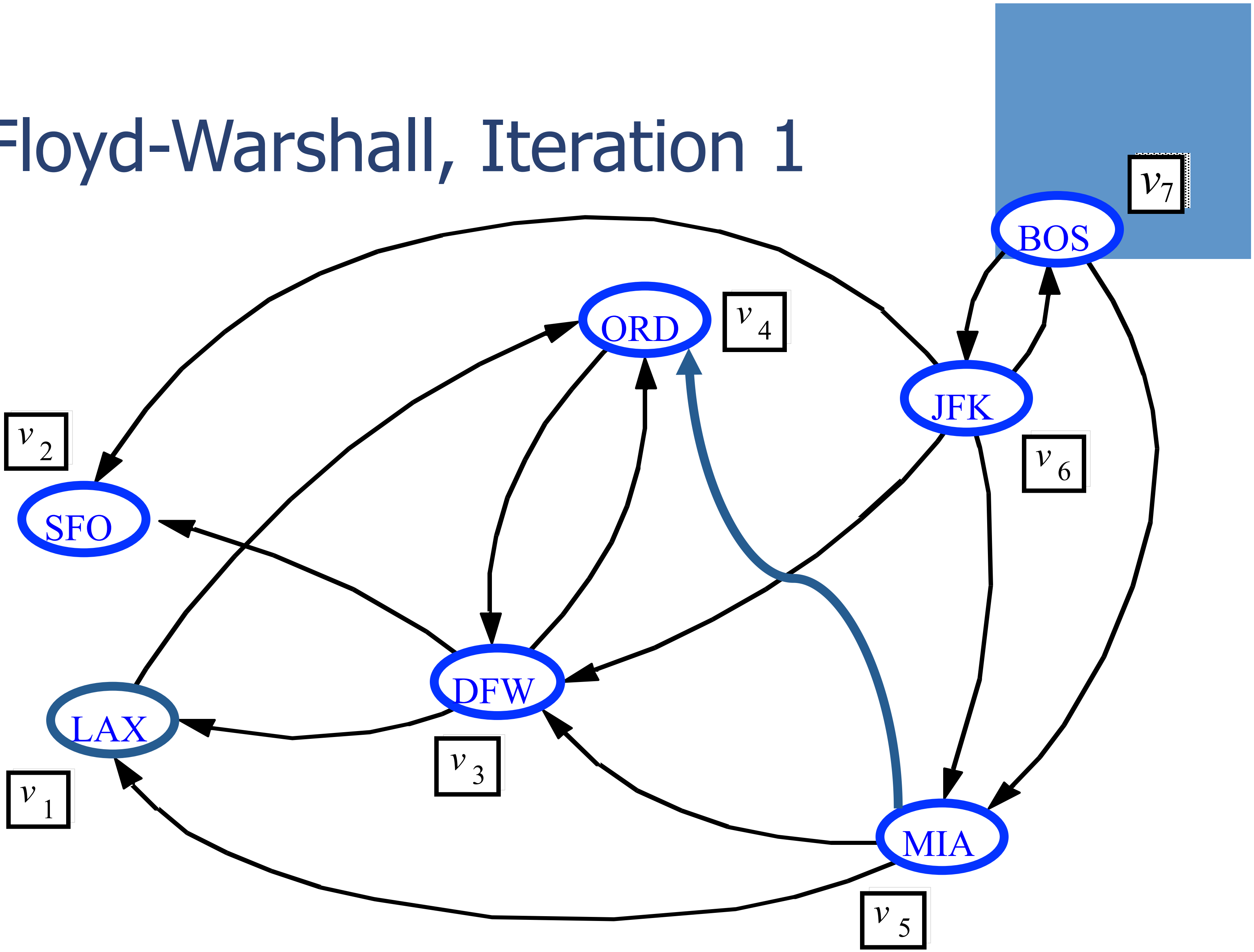
- Number vertices $v_1, \ldots, v_n$
- Compute digraphs $G_0, \ldots, G_n$
  - $G_0 = G$
  - G_k has directed edge (v_i, v_j) if G has a directed path from v_i to v_j with intermediate vertices in $\{v_1, \ldots, v_k\}$
- We have that $G_n = G^*$
- In phase k, digraph G_k is computed from G_{k-1}
- Running time: $O(n^3)$, assuming areAdjacent is $O(1)$ (e.g., adjacency matrix)

```
Algorithm FloydWarshall(G)
  Input digraph G
  Output transitive closure G* of G
  i ← 1
  for all v ∈ G.vertices()
    denote v as v_i
    i ← i + 1
  G_0 ← G
  for k ← 1 to n do
    G_k ← G_{k-1}
     for i ← 1 to n (i ≠ k) do
       for j ← 1 to n (j ≠ i, k) do
         if G_{k-1}.areAdjacent(v_i, v_k) ∧
               G_{k-1}.areAdjacent(v_k, v_j)
           if ¬ G_k.areAdjacent(v_i, v_j)
               G_k.insertDirectedEdge(v_i, v_j , k)
    return G_n
```

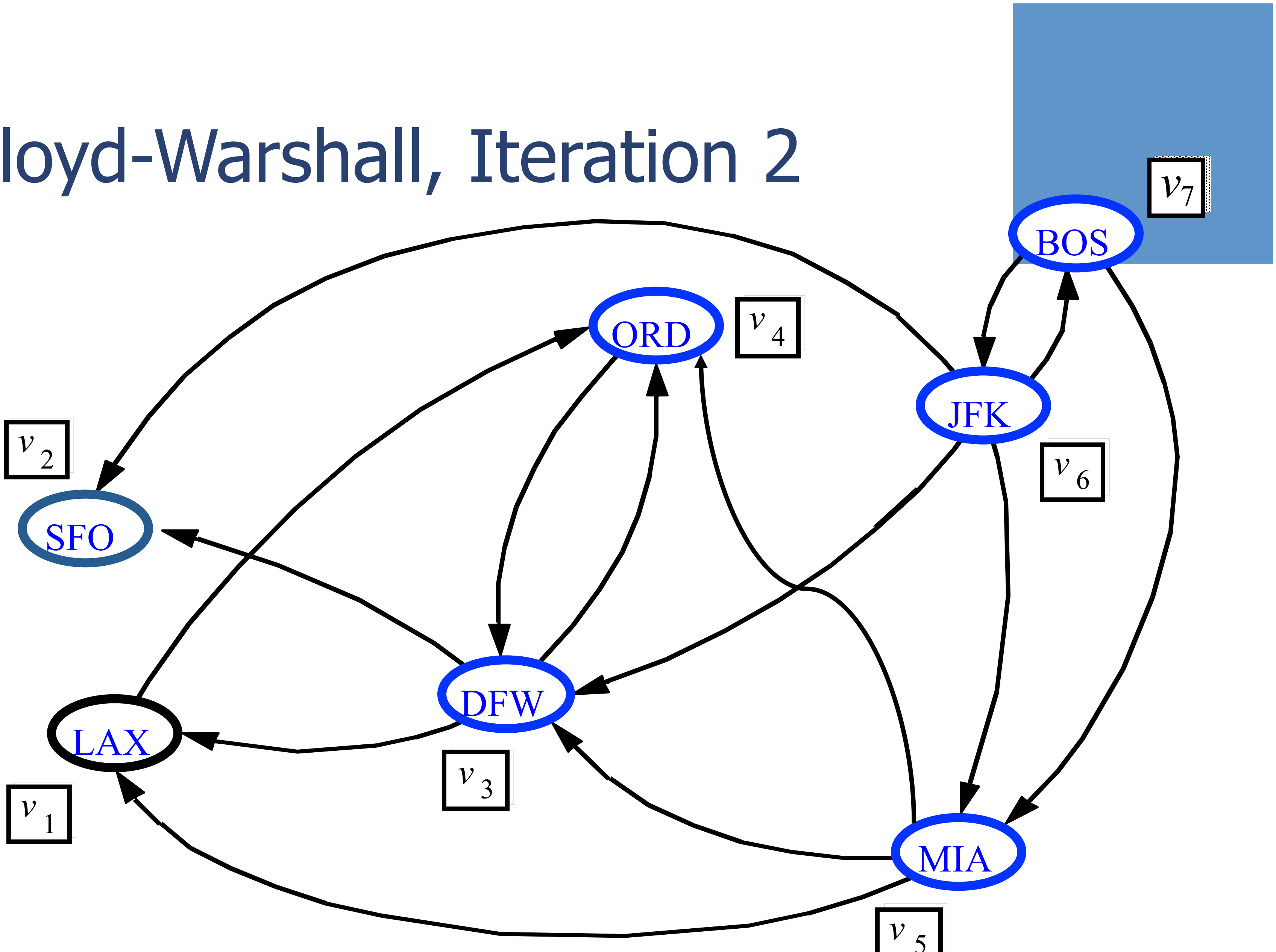
Floyd-Warshall Example



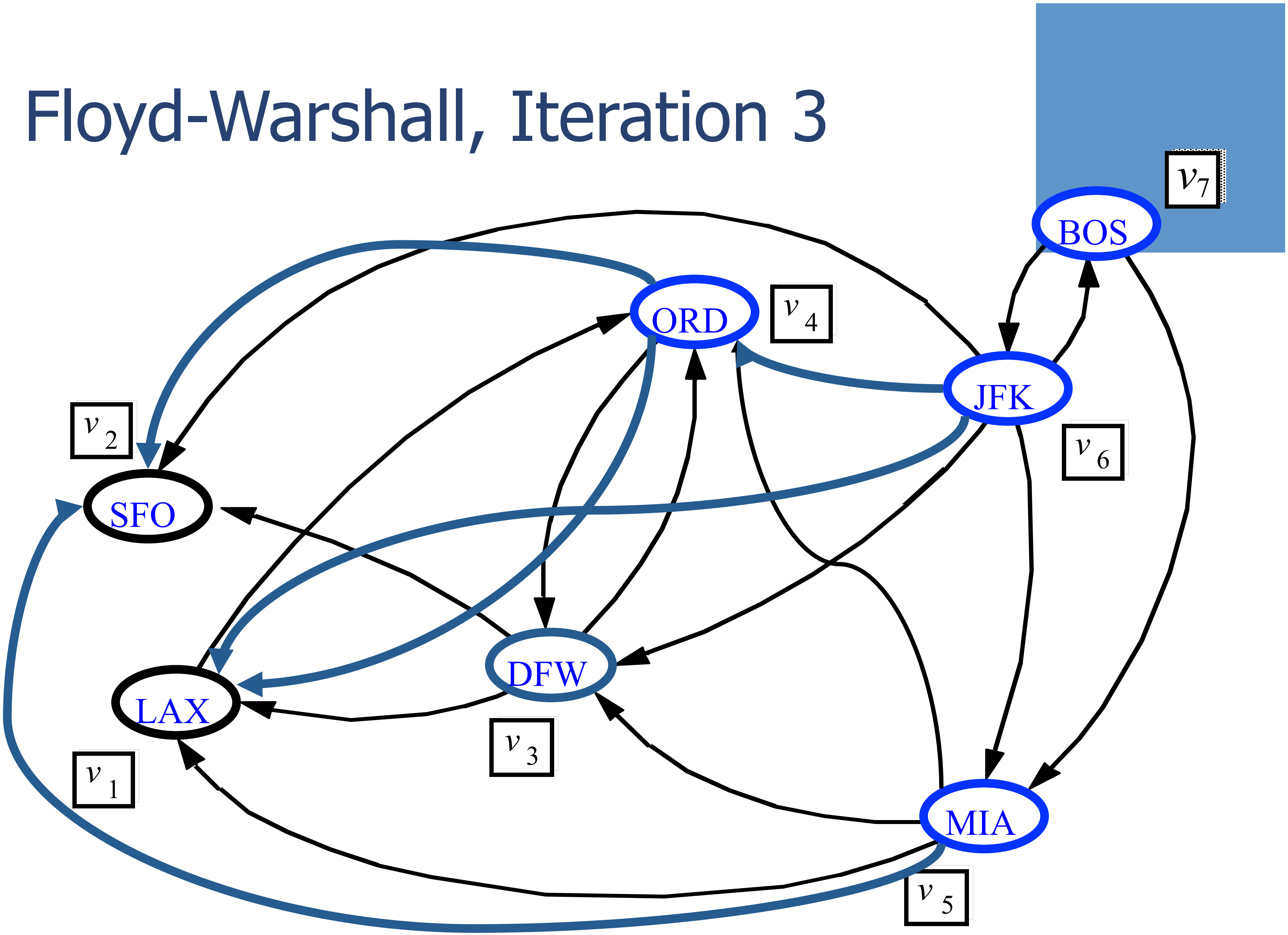
Floyd-Warshall, Iteration 1



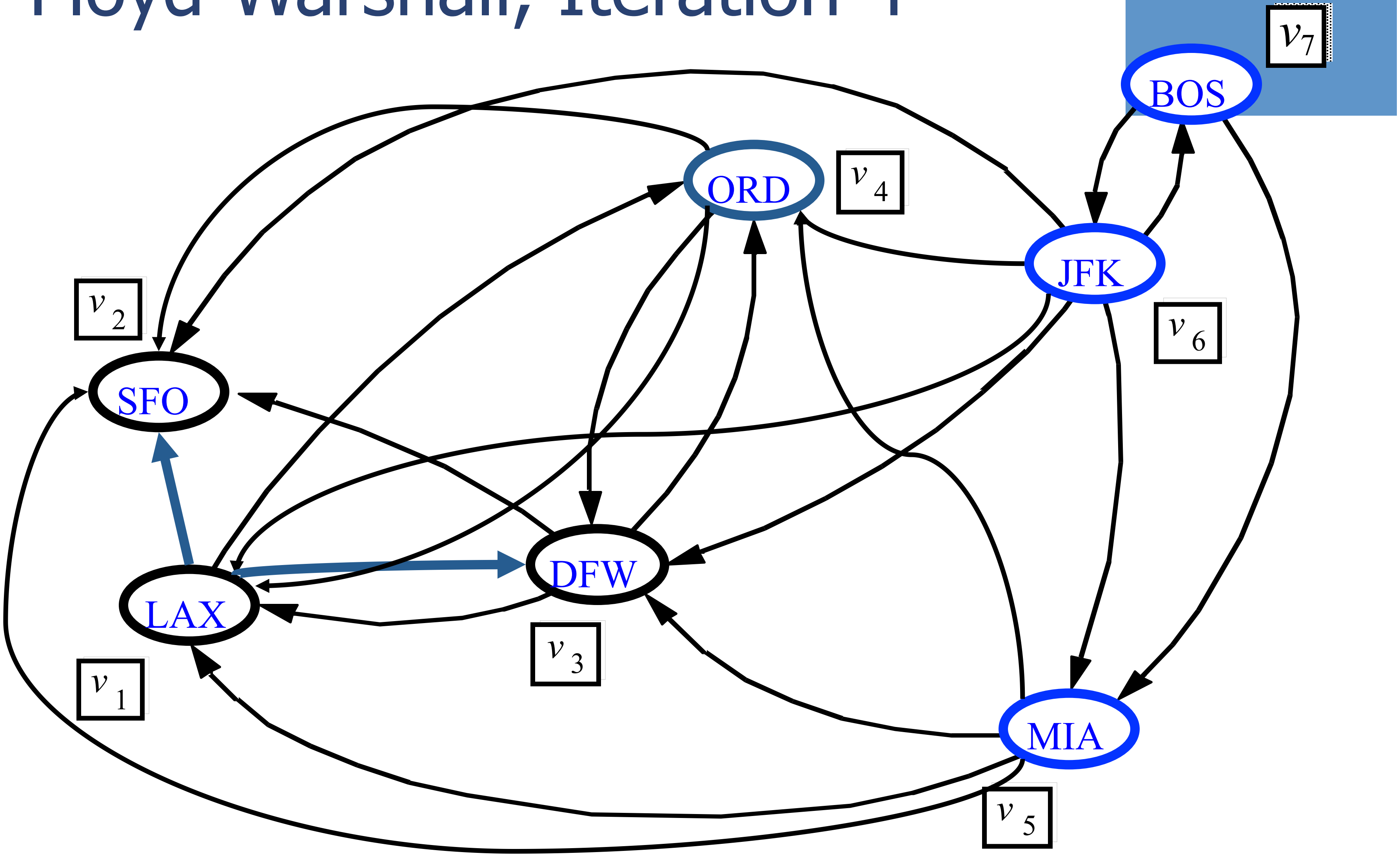
Floyd-Warshall, Iteration 2



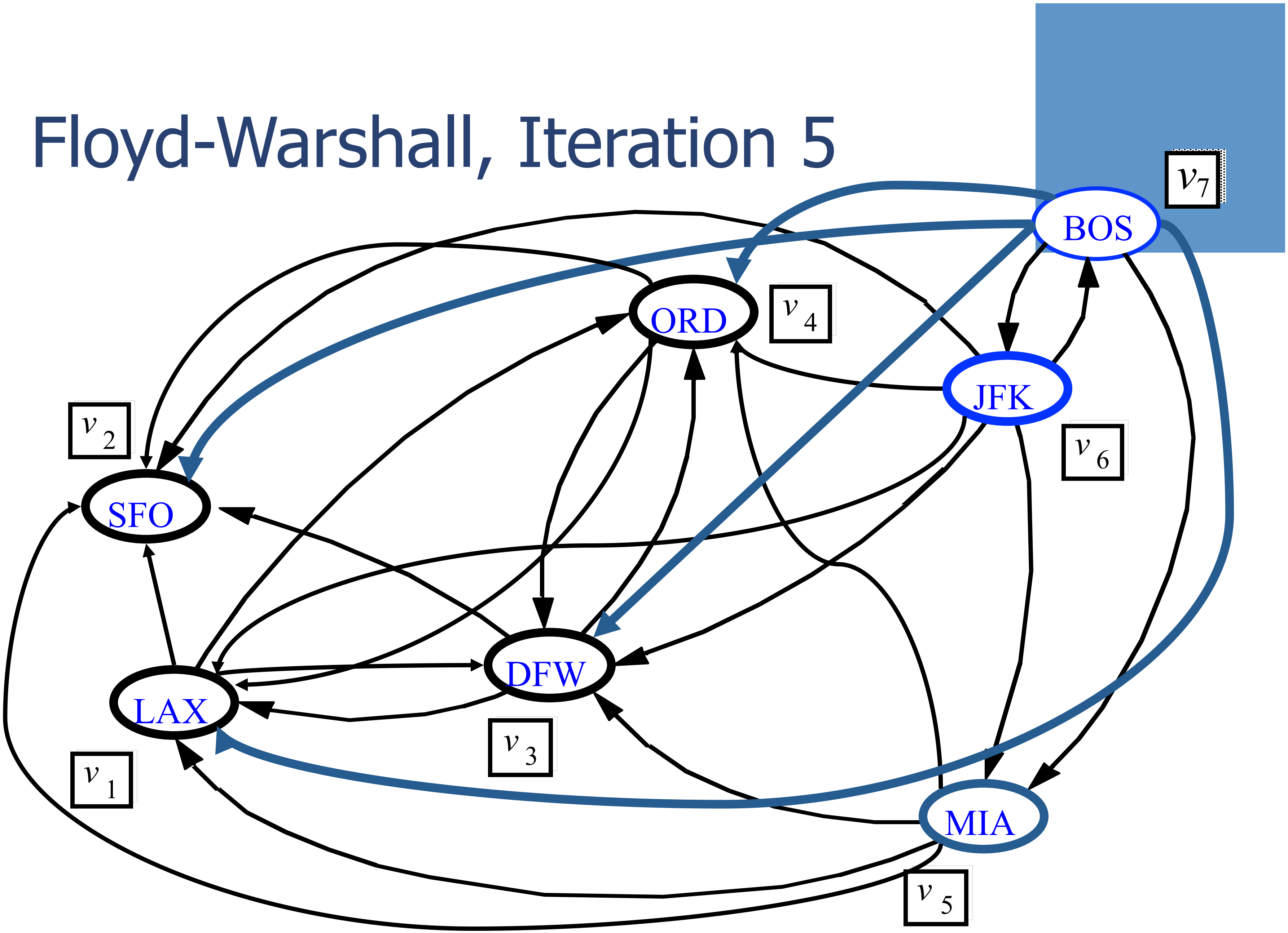
Floyd-Warshall, Iteration 3



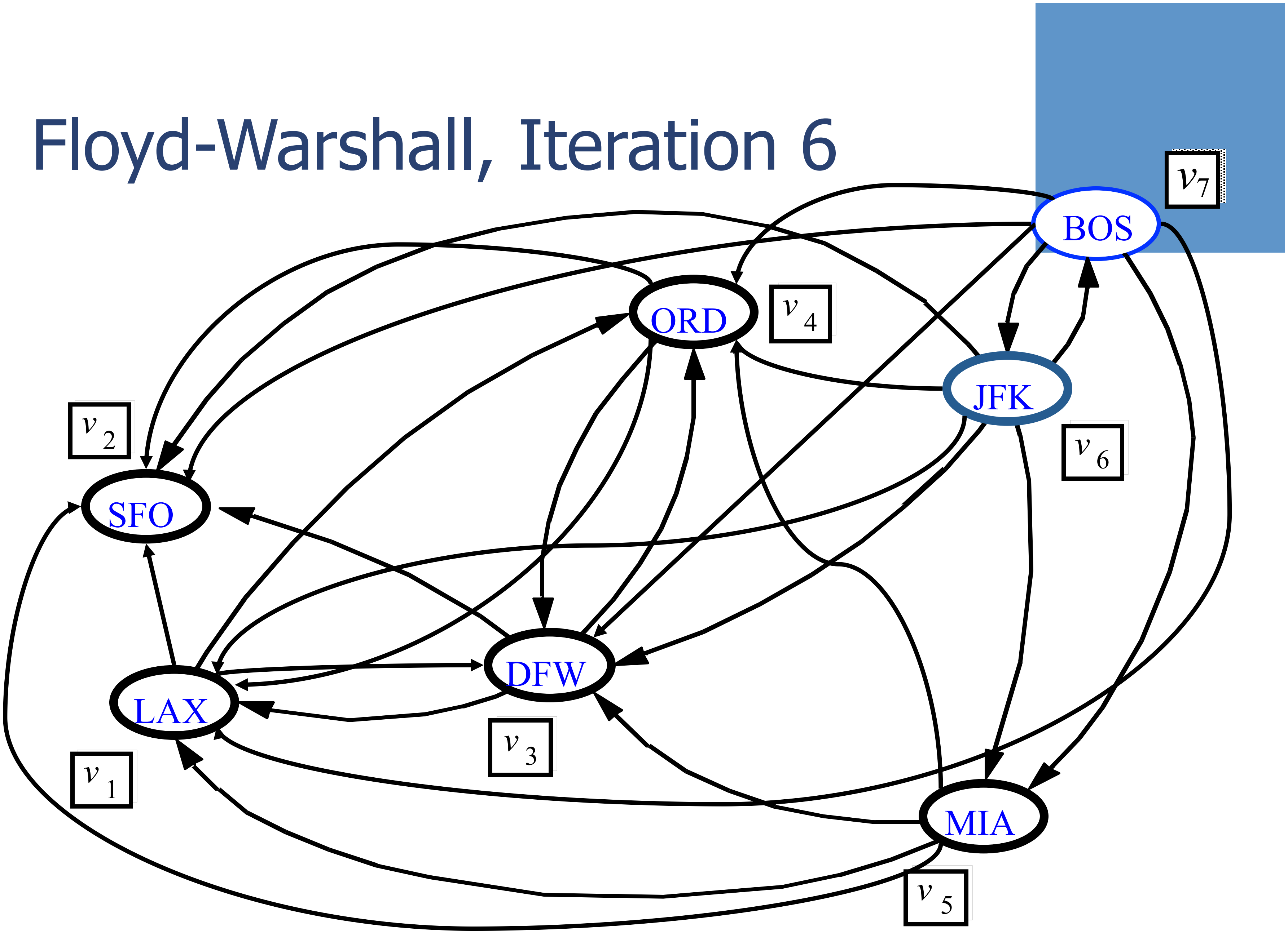
Floyd-Warshall, Iteration 4



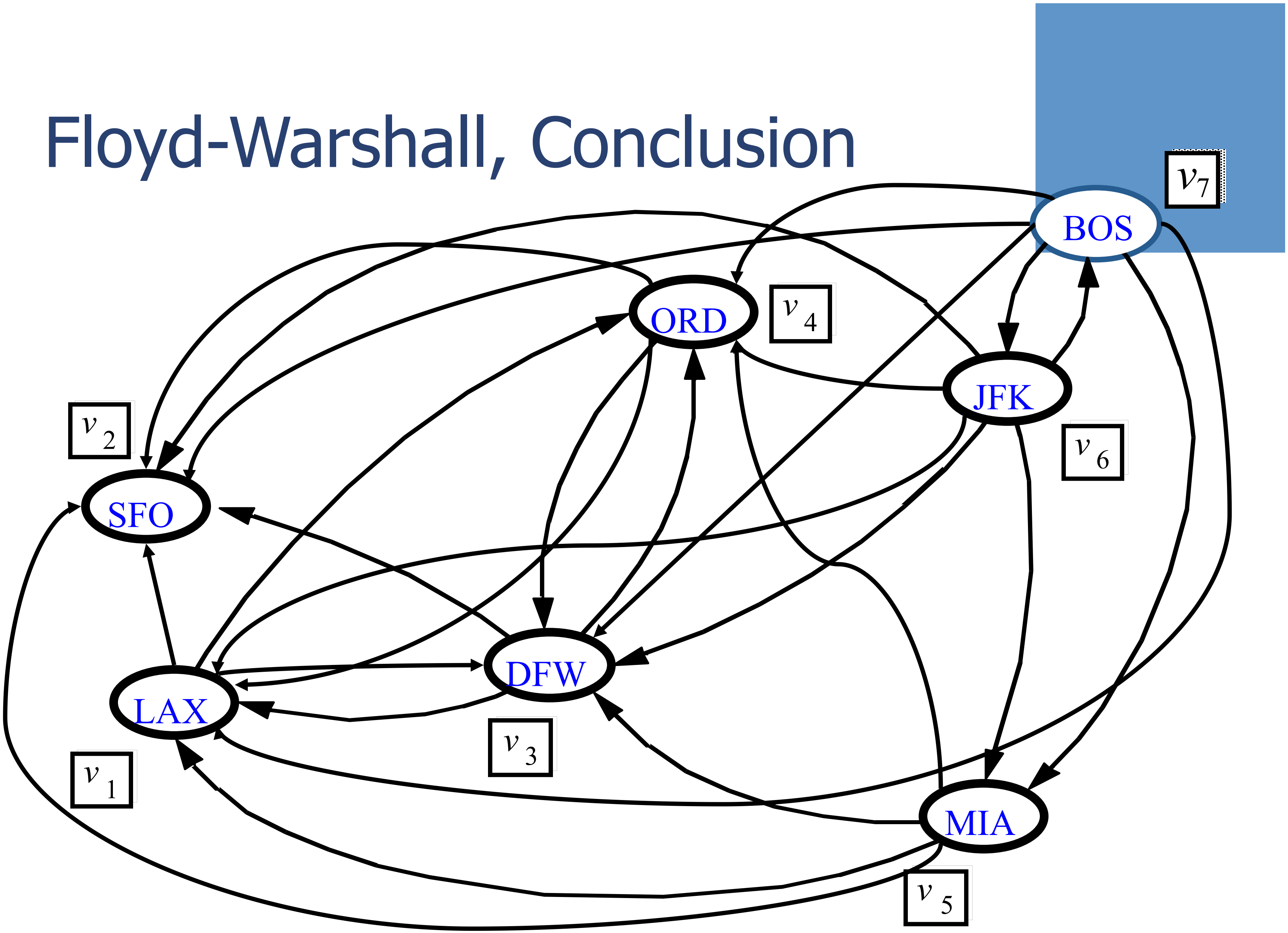
Floyd-Warshall, Iteration 5



Floyd-Warshall, Iteration 6

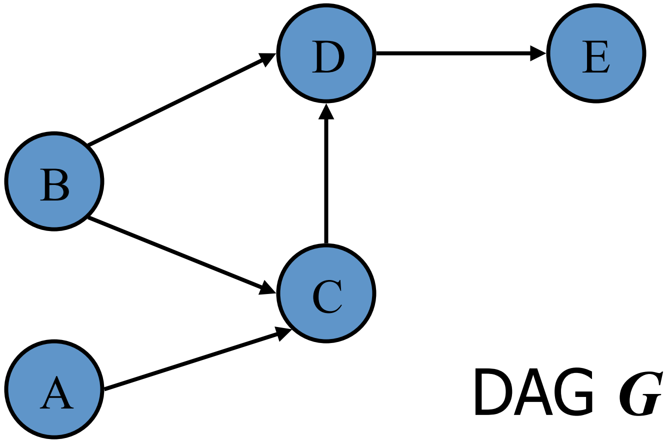


Floyd-Warshall, Conclusion



DAGs and Topological Ordering

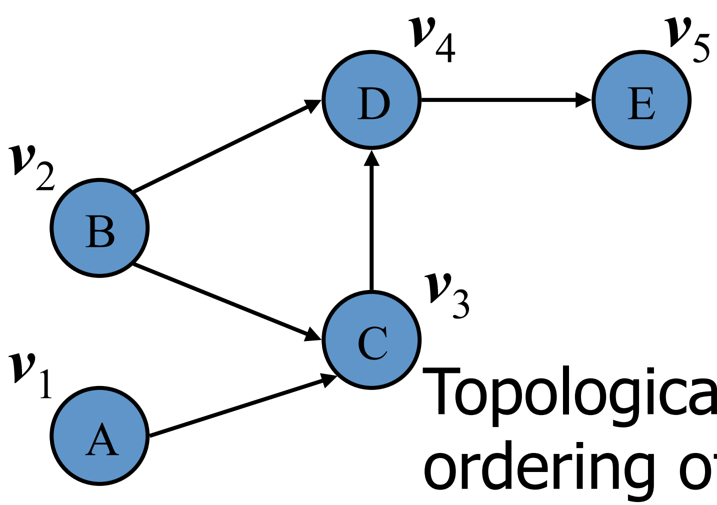
- A directed acyclic graph (DAG) is a digraph that has no directed cycles



DAG G

DAGs and Topological Ordering

- A topological ordering of a digraph is a numbering $v_1, \ldots, v_n$ of the vertices such that for every edge (v_i, v_j), we have $i < j$
- Example: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints



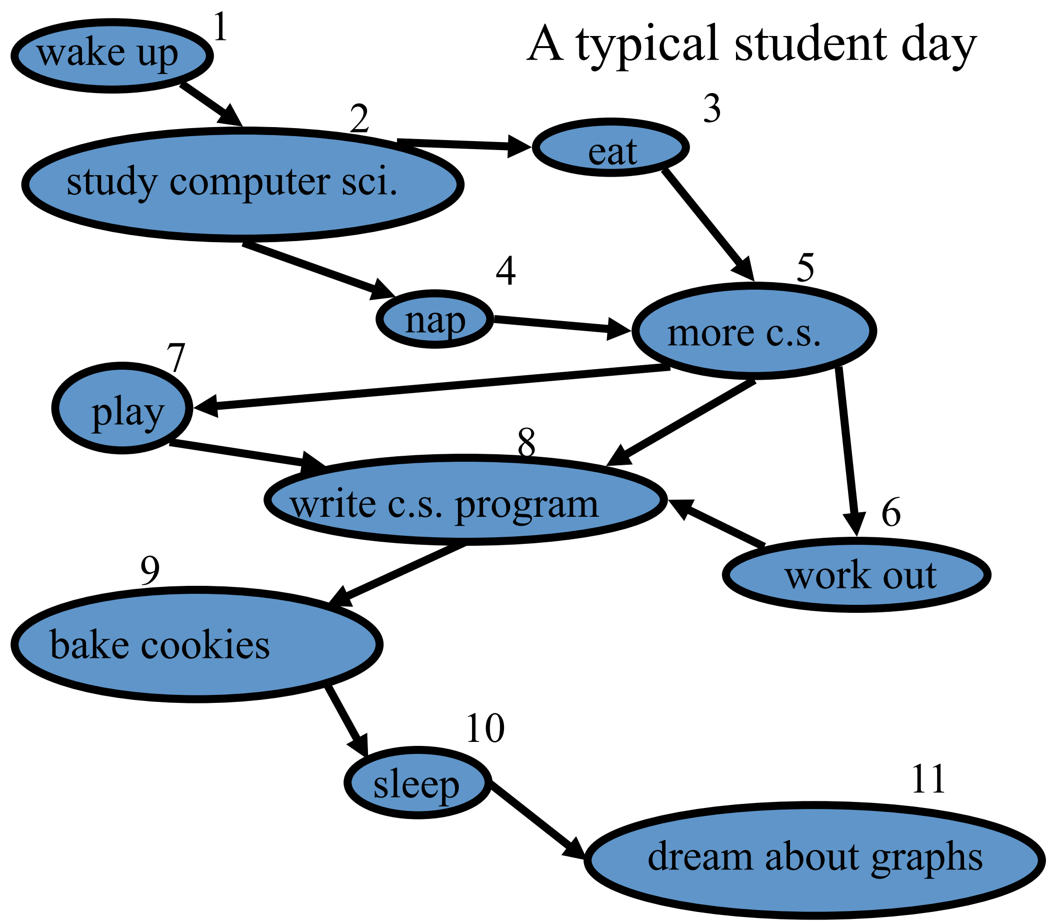
Topological ordering of G

Theorem

A digraph admits a topological ordering if and only if it is a DAG

Topological Sorting

- Number vertices, so that (u,v) in E implies u < v



Algorithm for Topological Sorting

- Note: This algorithm is different than the one in the book

Algorithm TopologicalSort(G)
  $H \leftarrow G$ // Temporary copy of G
  $n \leftarrow G.numVertices()$
  while H is not empty **do**
    Let v be a vertex with no outgoing edges
    Label $v \leftarrow n$
    $n \leftarrow n - 1$
    Remove v from H

- Running time: O(n + m)

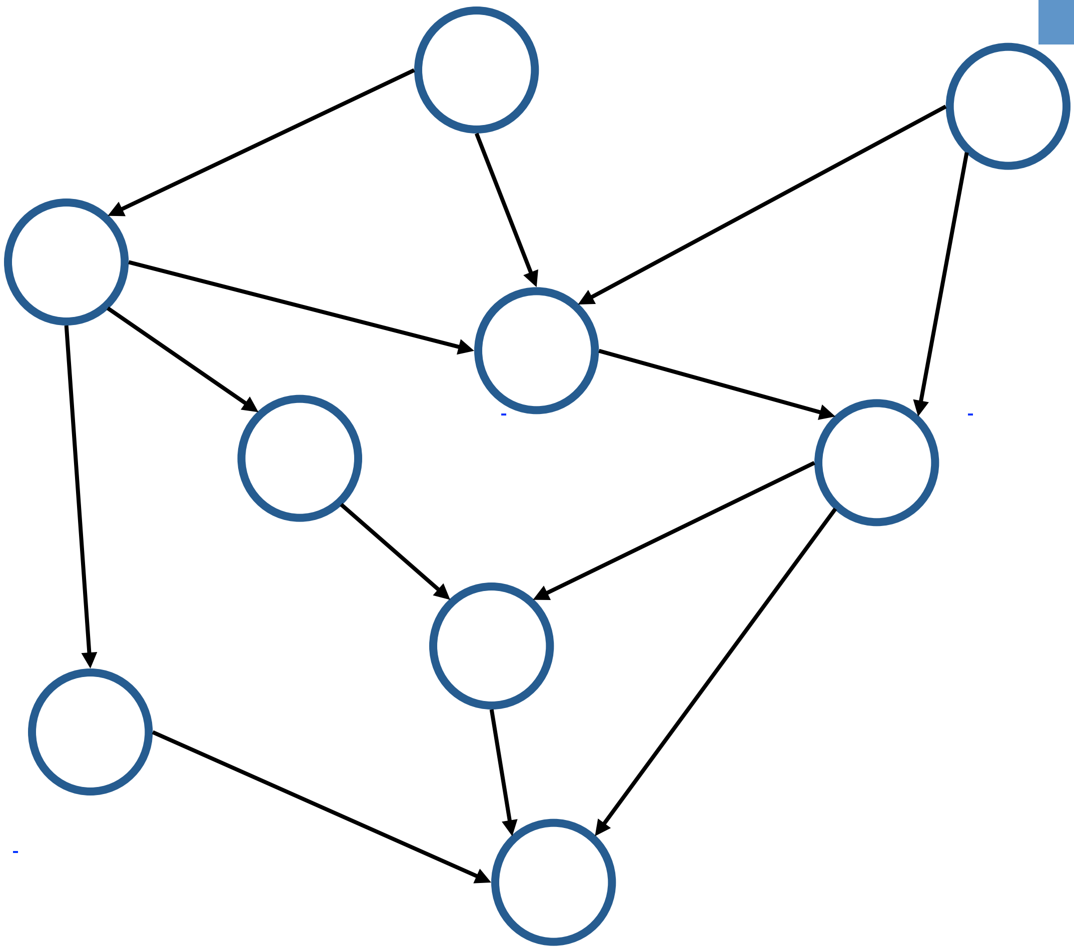
Implementation with DFS

- Simulate the algorithm by using depth-first search
- O(n+m) time.

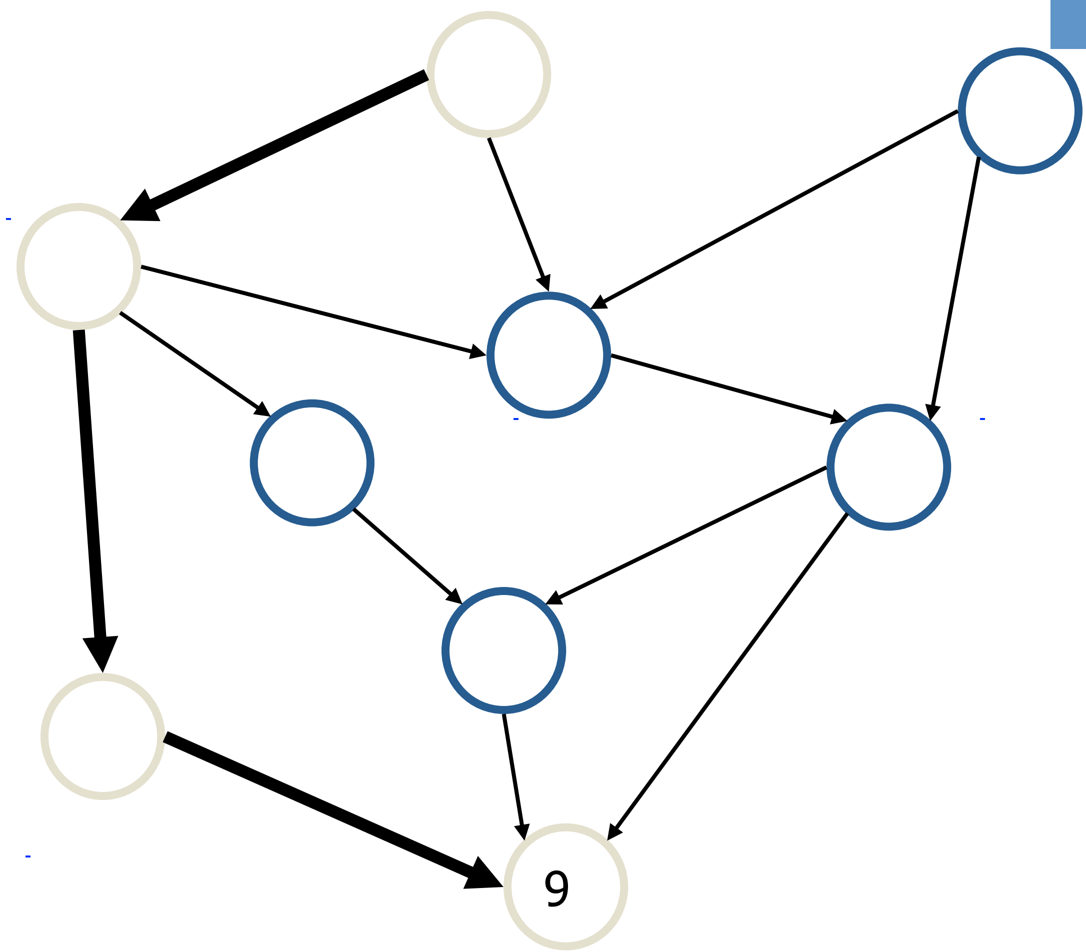
Algorithm *topologicalDFS*(G)
  Input dag G
  Output topological ordering of G
 $n \leftarrow G.numVertices()$
 for all $u \in G.vertices()$
  $setLabel(u, UNEXPLORED)$
 for all $v \in G.vertices()$
  if $getLabel(v) = UNEXPLORED$
   $topologicalDFS(G, v)$

Algorithm *topologicalDFS*(G, v)
 Input graph G and a start vertex v of G
 Output labeling of the vertices of G
  in the connected component of v
 $setLabel(v, VISITED)$
 for all $e \in G.outEdges(v)$
  { outgoing edges }
  $w \leftarrow opposite(v,e)$
  if $getLabel(w) = UNEXPLORED$
   { e is a discovery edge }
   $topologicalDFS(G, w)$
  else
   { e is a forward or cross edge }
 Label v with topological number n
 $n \leftarrow n - 1$

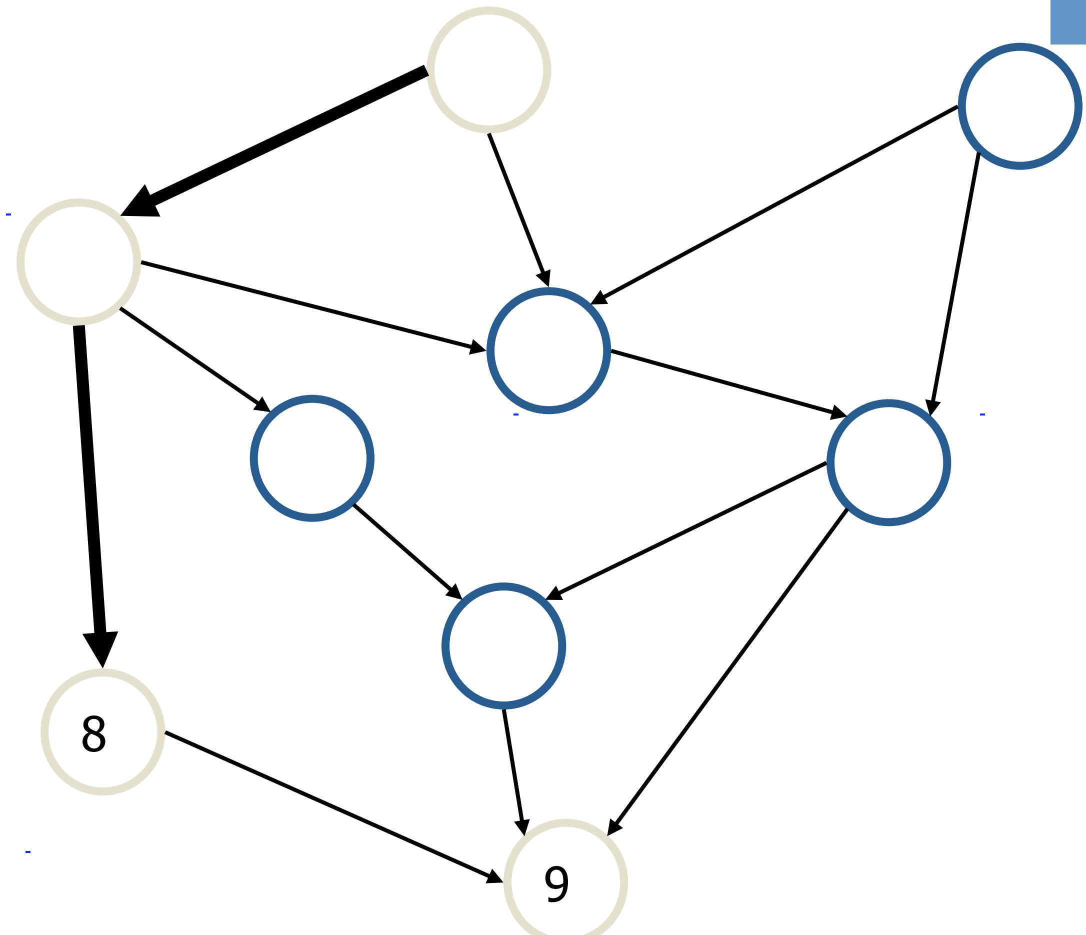
Topological Sorting Example



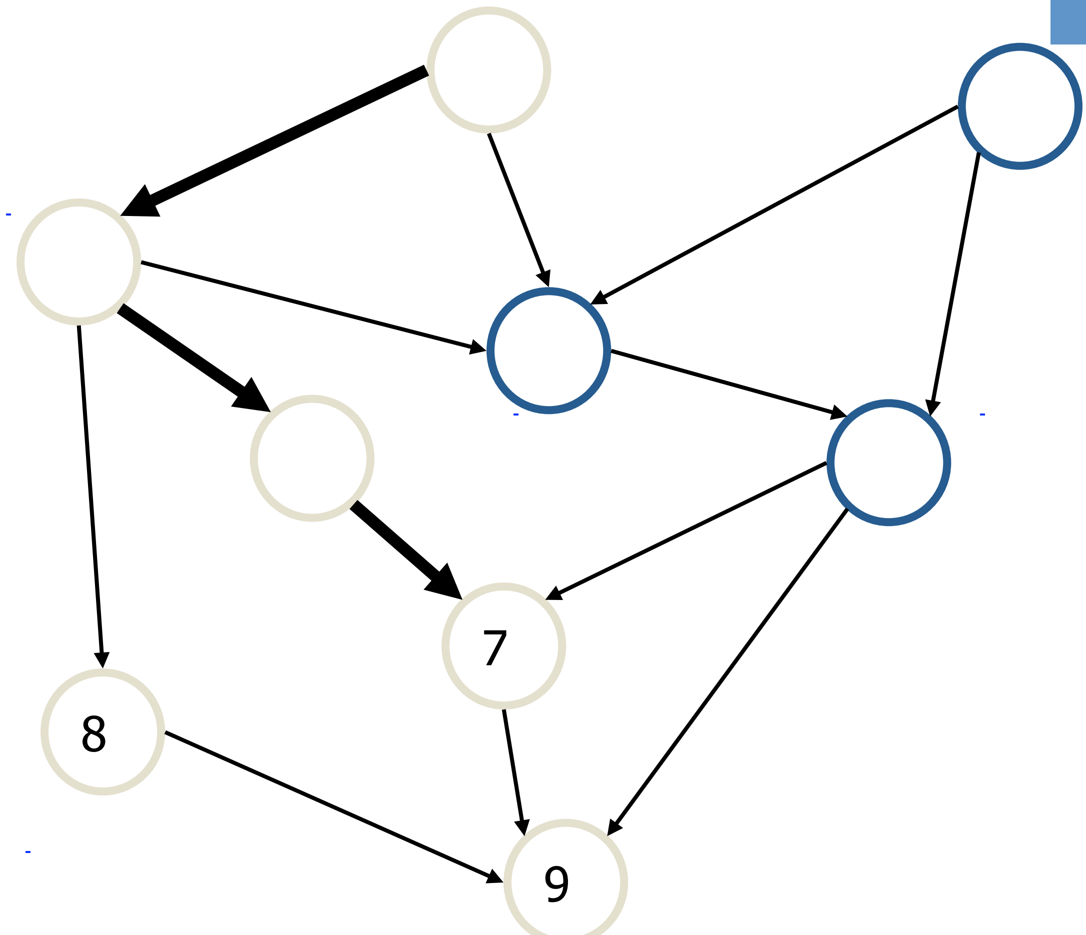
Topological Sorting Example



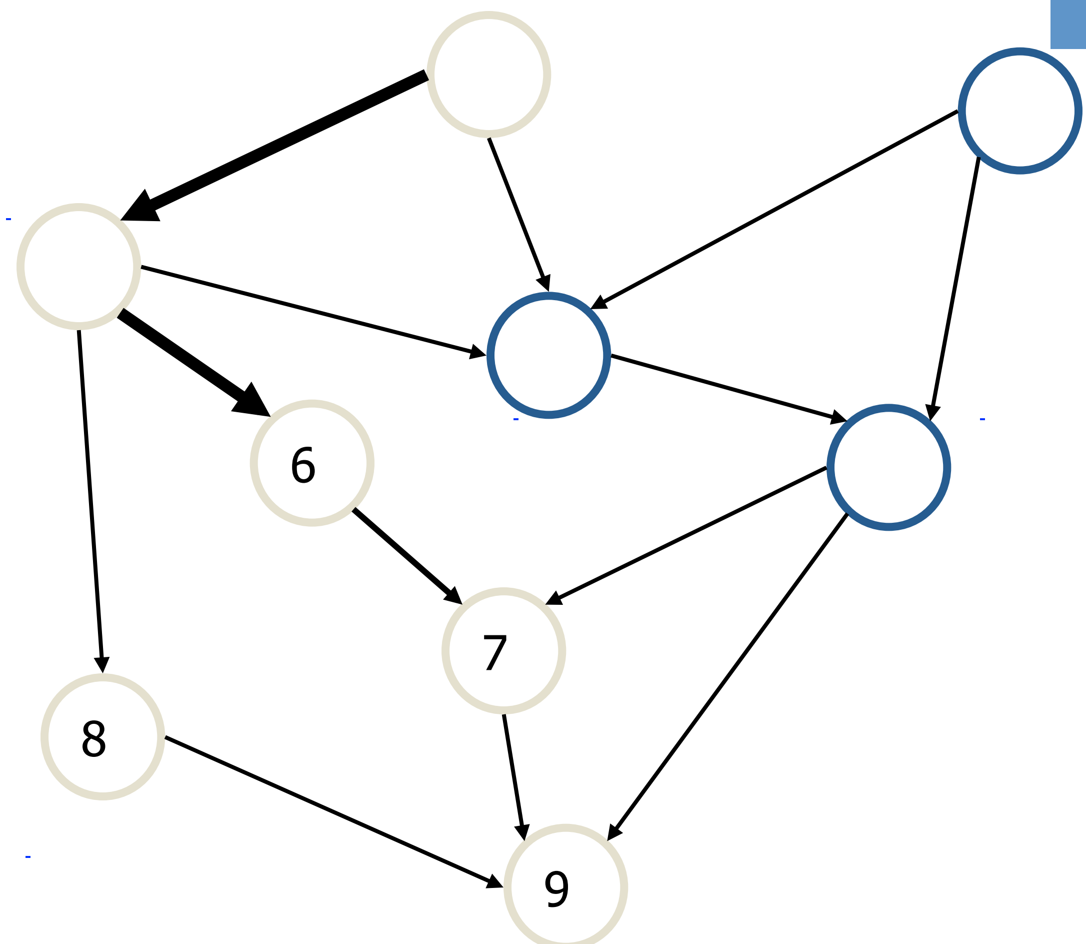
Topological Sorting Example



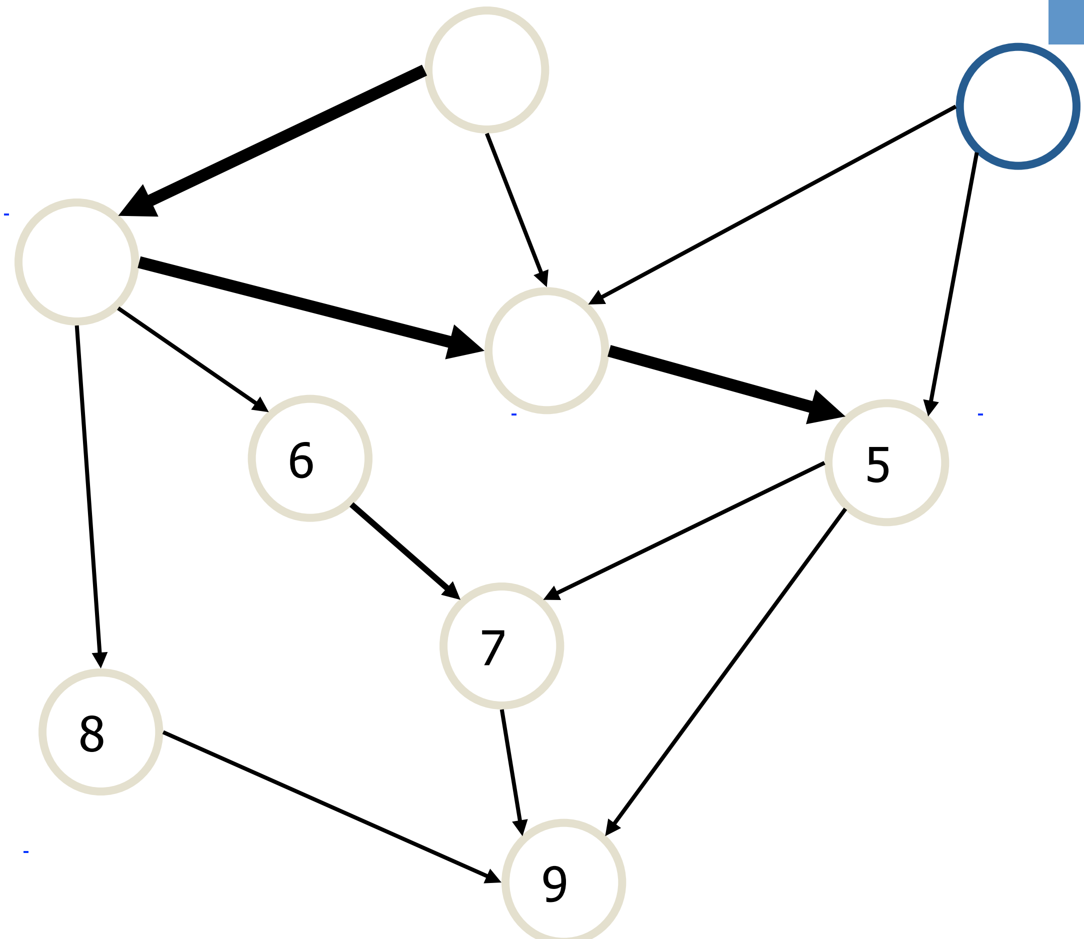
Topological Sorting Example



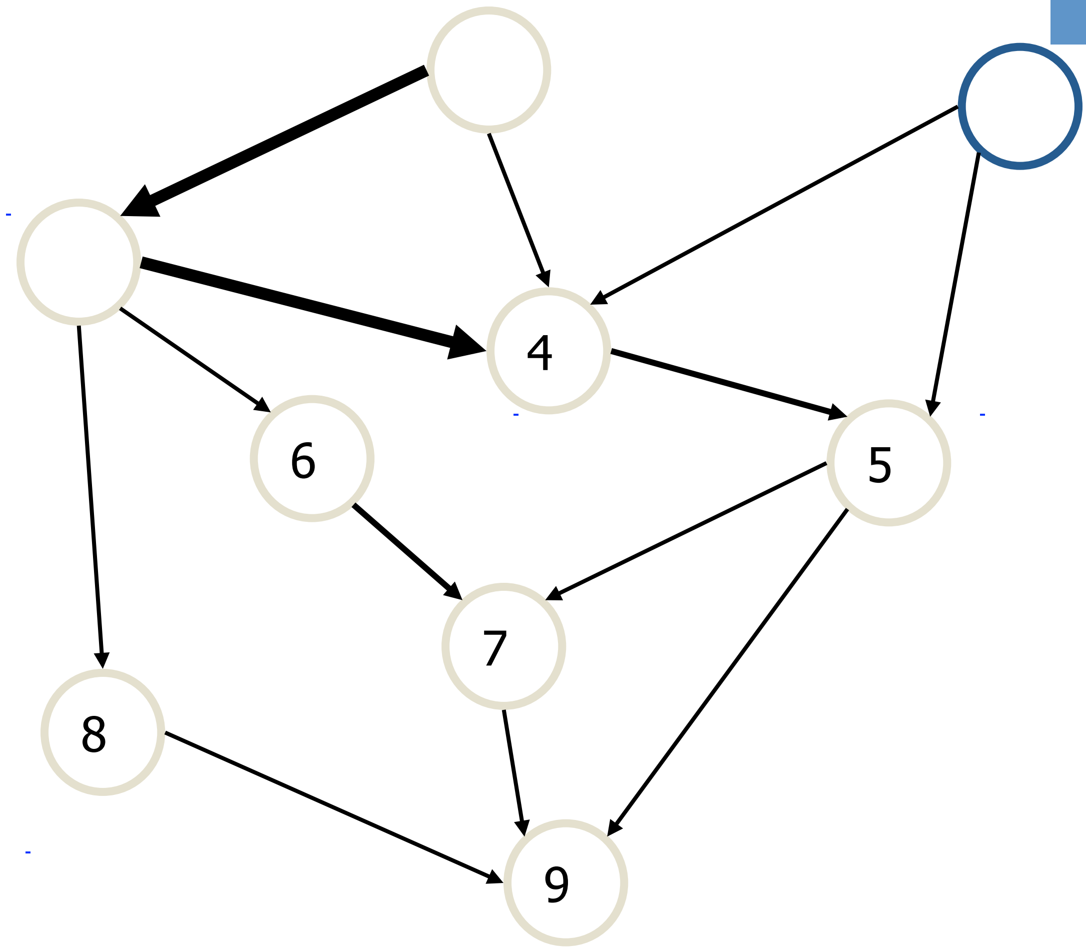
Topological Sorting Example



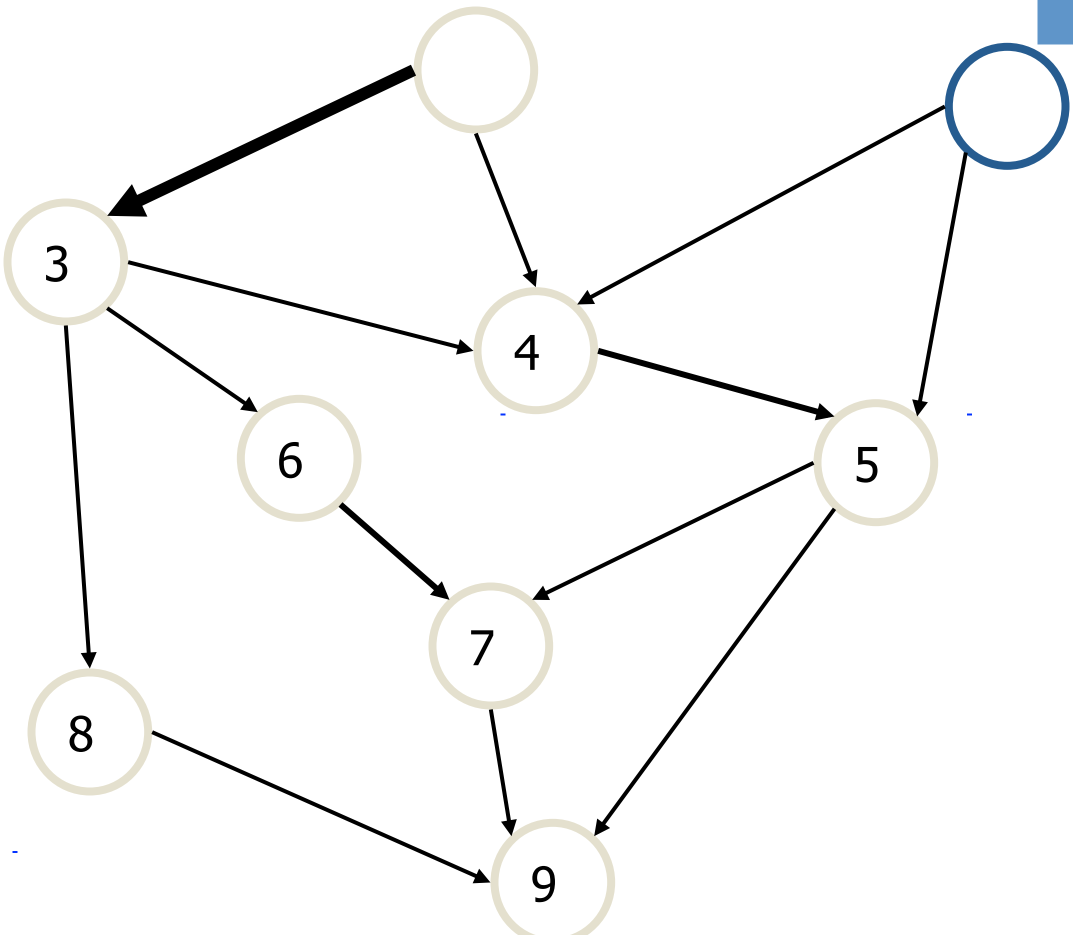
Topological Sorting Example



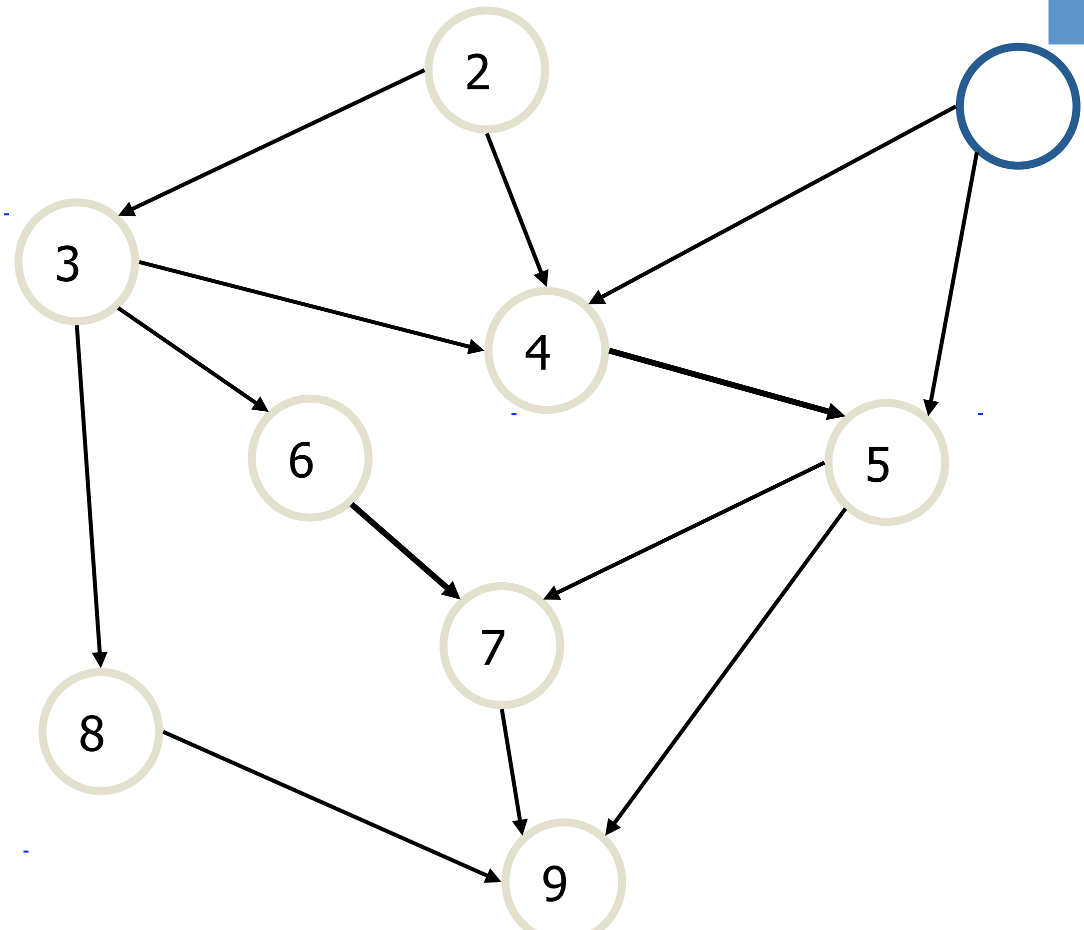
Topological Sorting Example



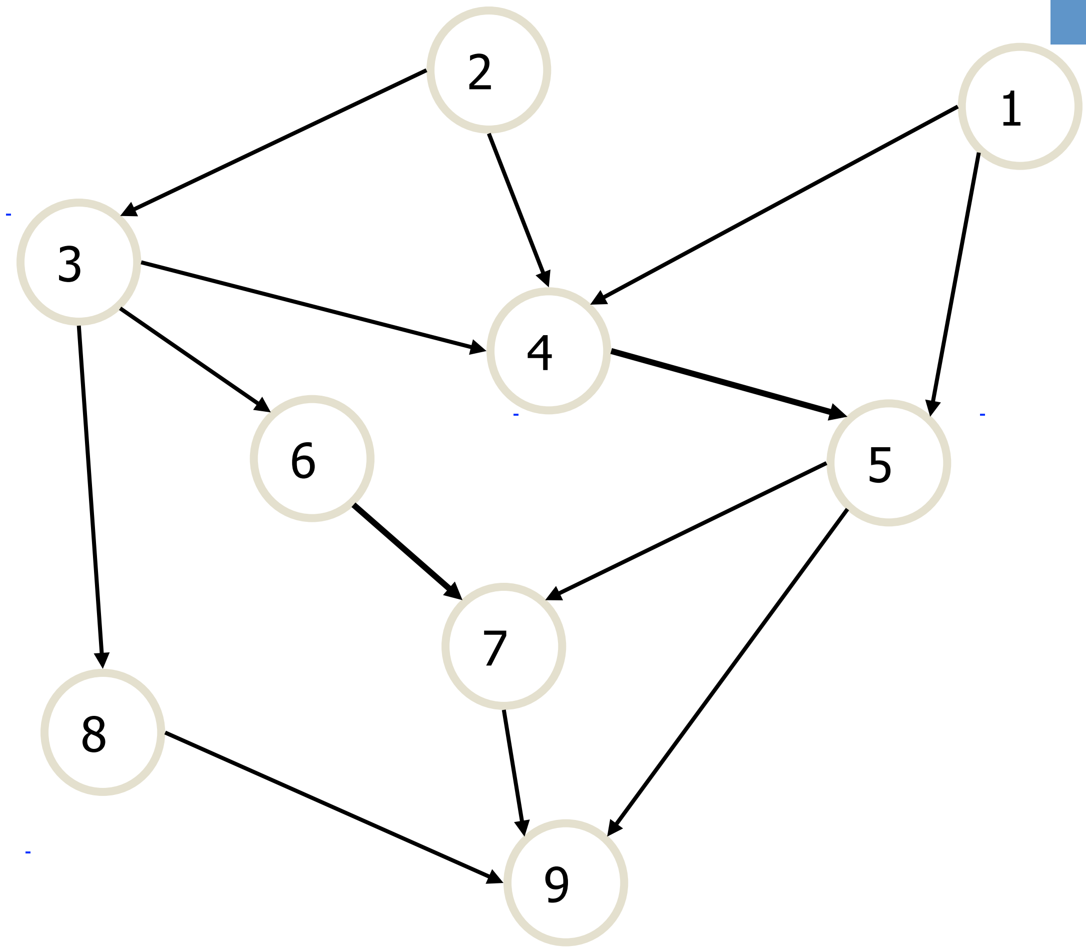
Topological Sorting Example



Topological Sorting Example



Topological Sorting Example



A Quiz

- Fang loves CS courses and wants to plan his course schedule. The course prerequisites are:
  - CS15: (none)
  - CS16: CS15
  - CS22: (none)
  - CS31: CS15
  - CS32: CS16, CS31
  - CS126: CS22, CS32, CS16
  - CS127: CS16
  - CS141: CS22, CS16
  - CS169:CS32

Please help Fang to find the sequence of courses that allows him to satisfy all the prerequisities.

Minimum Spanning Trees

Spanning subgraph

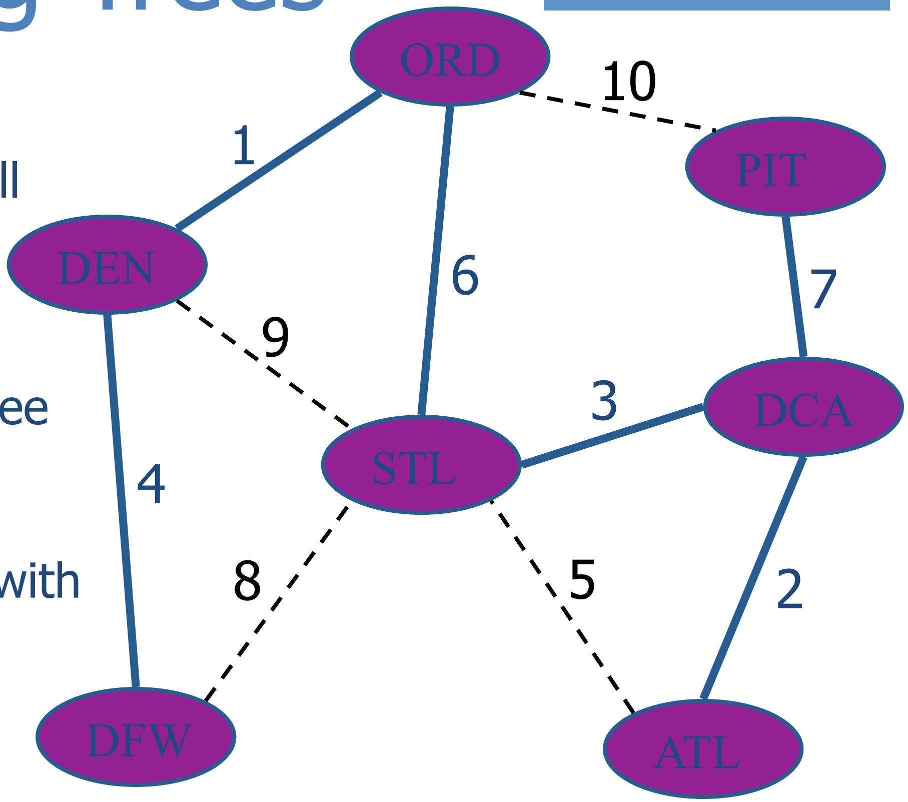
- Subgraph of a graph G containing all the vertices of G

Spanning tree

- Spanning subgraph that is itself a tree

Minimum spanning tree (MST)

- Spanning tree of a weighted graph with minimum total edge weight



- Applications
  - Communications networks
  - Transportation networks

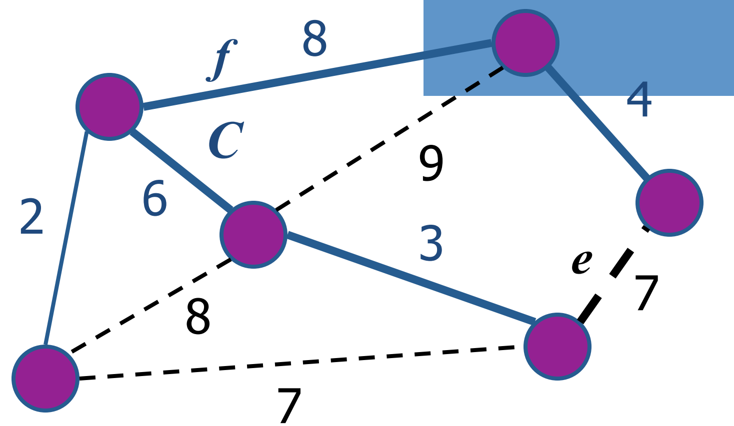
Cycle Property

Cycle Property:

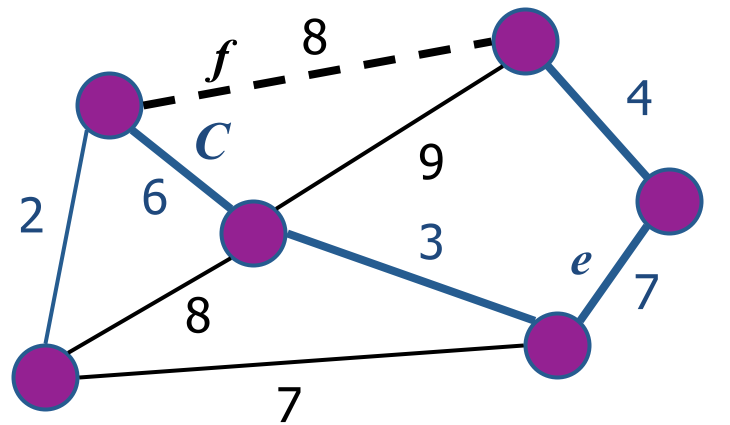
- Let T be a minimum spanning tree of a weighted graph G
- Let e be an edge of G that is not in T and C let be the cycle formed by e with T
- For every edge f of C, $weight(f) \leq weight(e)$

Proof:

- By contradiction
- If $weight(f) > weight(e)$ we can get a spanning tree of smaller weight by replacing e with f

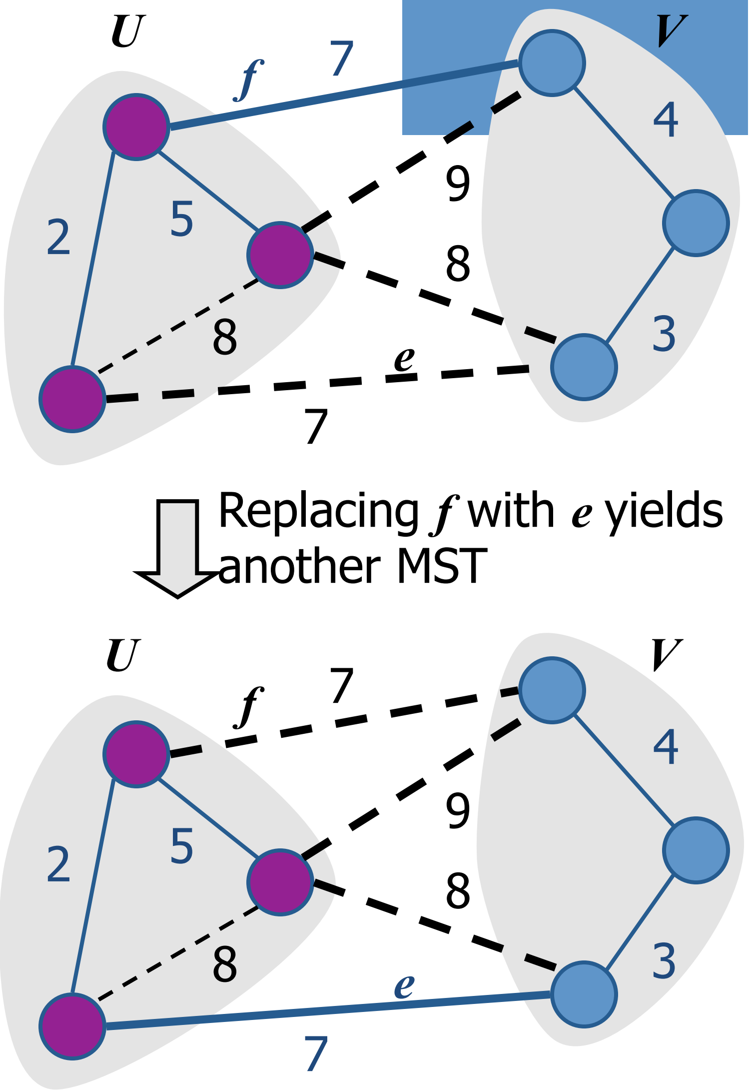


Replacing f with e yields a better spanning tree



Partition Property

- Partition Property:
  - Consider a partition of the vertices of G into subsets U and V
  - Let e be an edge of minimum weight across the partition
  - There is a minimum spanning tree of G containing edge e

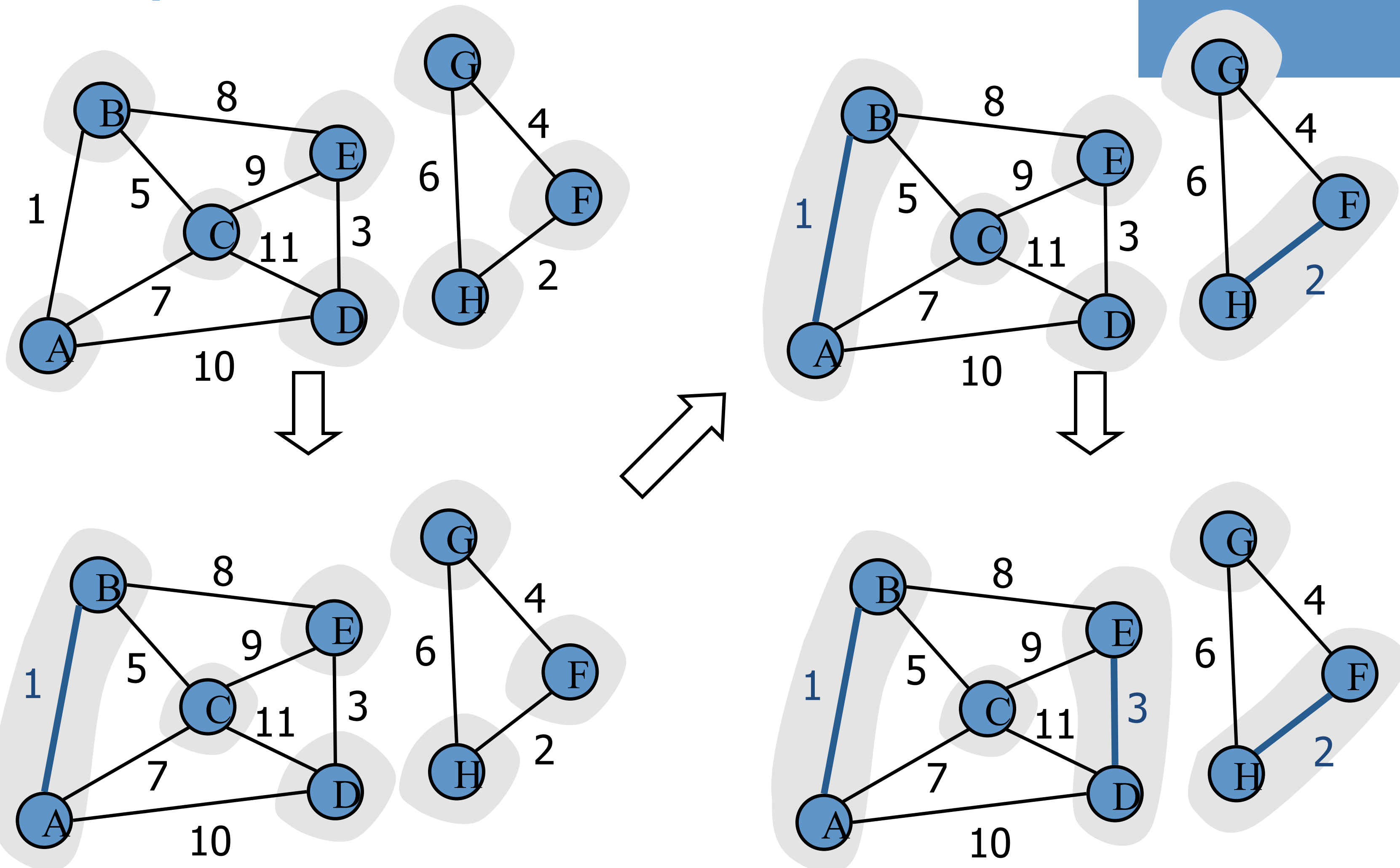


Kruskal's Algorithm

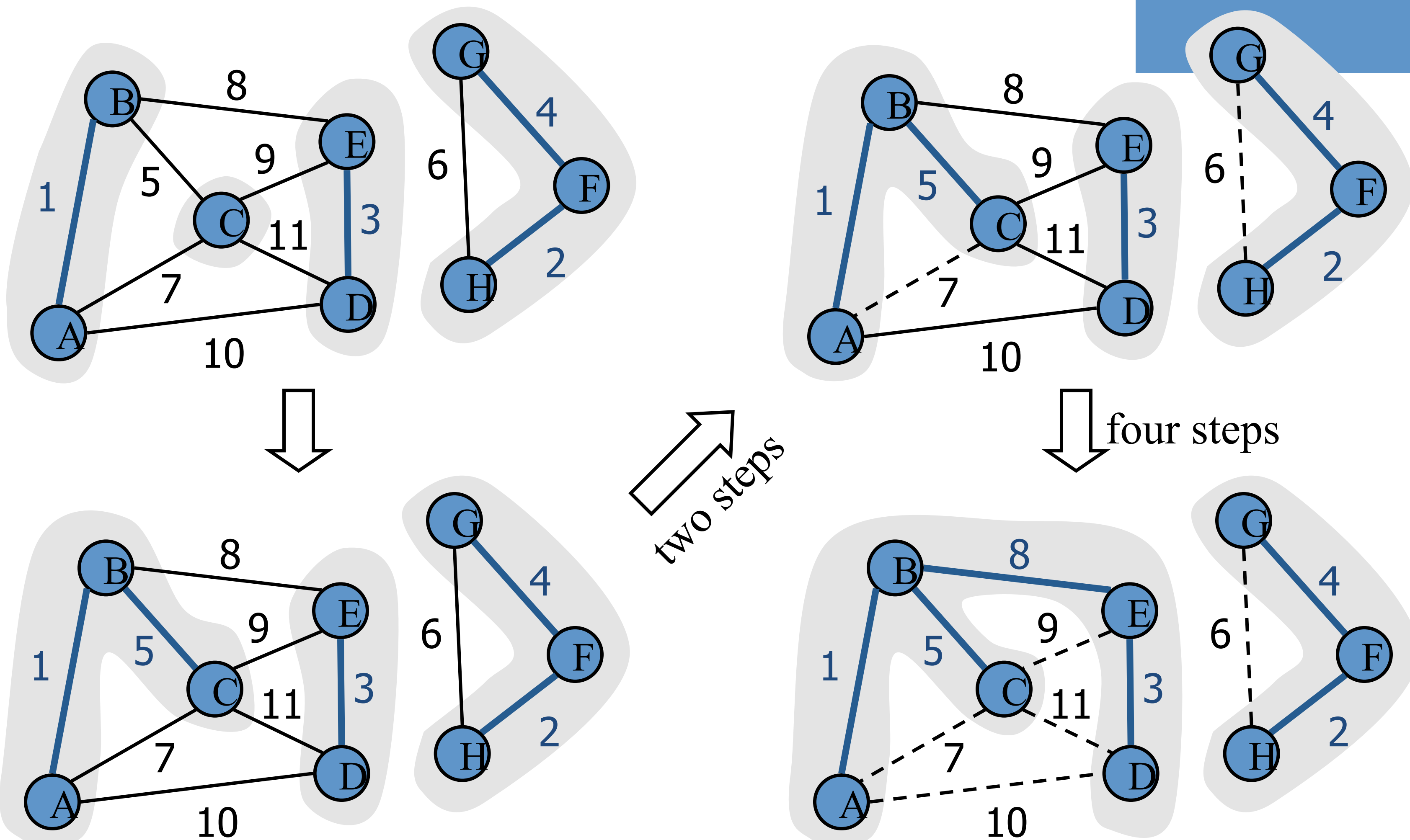
- Maintain a partition of the vertices into clusters
  - Initially, single-vertex clusters
  - Keep an MST for each cluster
  - Merge "closest" clusters and their MSTs
- A priority queue stores the edges outside clusters
  - Key: weight
  - Element: edge
- At the end of the algorithm
  - One cluster and one MST

Algorithm *KruskalMST*(G)
    for each vertex v in G **do**
        Create a cluster consisting of v
    let Q be a priority queue.
    Insert all edges into Q
    $T \leftarrow \emptyset$
    {T is the union of the MSTs of the clusters}
    while T has fewer than $n - 1$ edges **do**
        $e \leftarrow$ *Q.removeMin().getValue*()
        $[u, v] \leftarrow$ *G.endVertices*(e)
        $A \leftarrow$ *getCluster*(u)
        $B \leftarrow$ *getCluster*(v)
        if $A \neq B$ **then**
            Add edge e to T
            mergeClusters(A, B)
    return T

Example



Example (contd.)



Prim-Jarnik's Algorithm

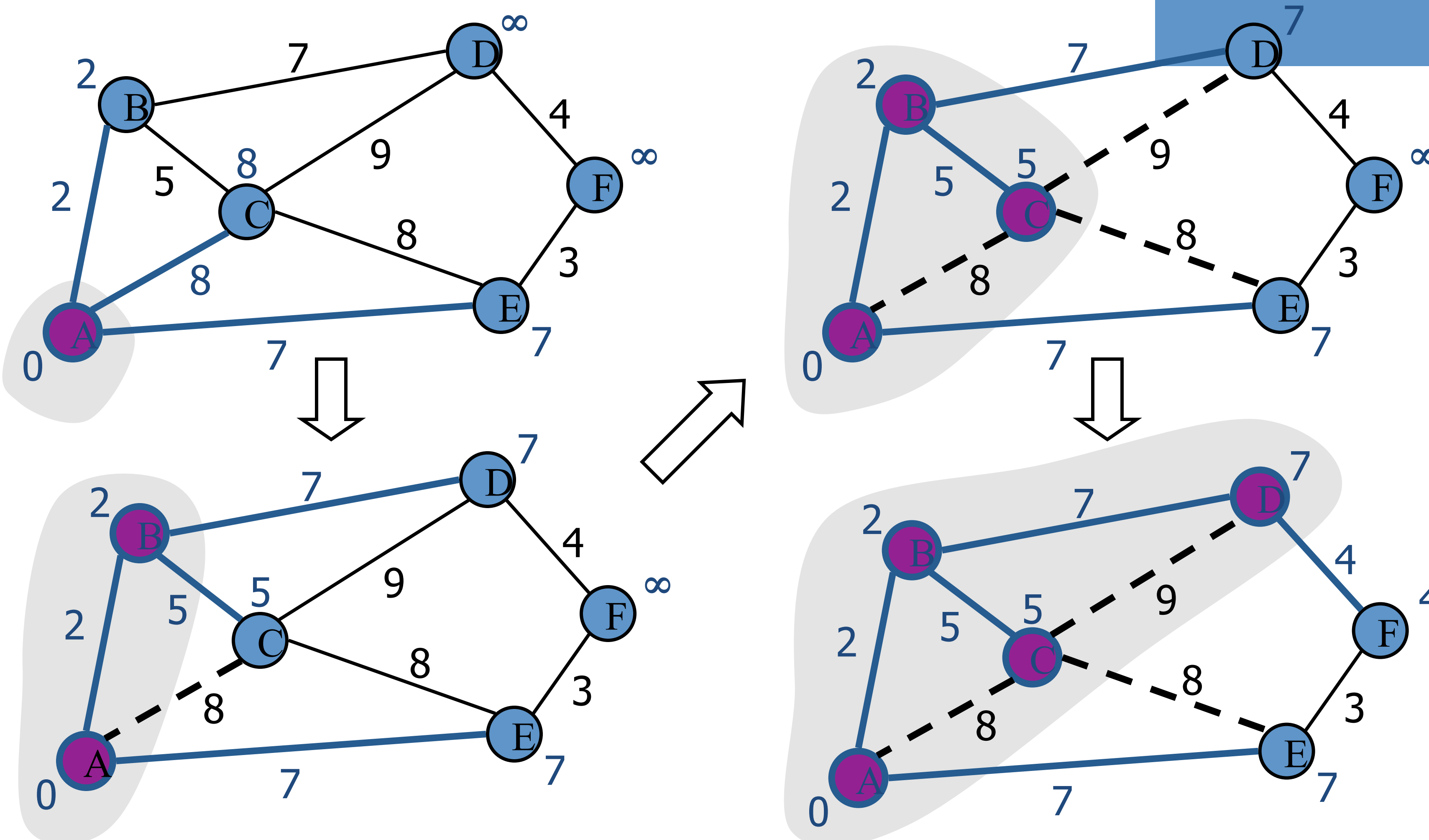
- We pick an arbitrary vertex s and we grow the MST as a cloud of vertices, starting from s
- We store with each vertex v label $d(v)$ representing the smallest weight of an edge connecting v to a vertex in the cloud
- At each step:
  - We add to the cloud the vertex u outside the cloud with the smallest distance label
  - We update the labels of the vertices adjacent to u

Prim-Jarnik's Algorithm (cont.)

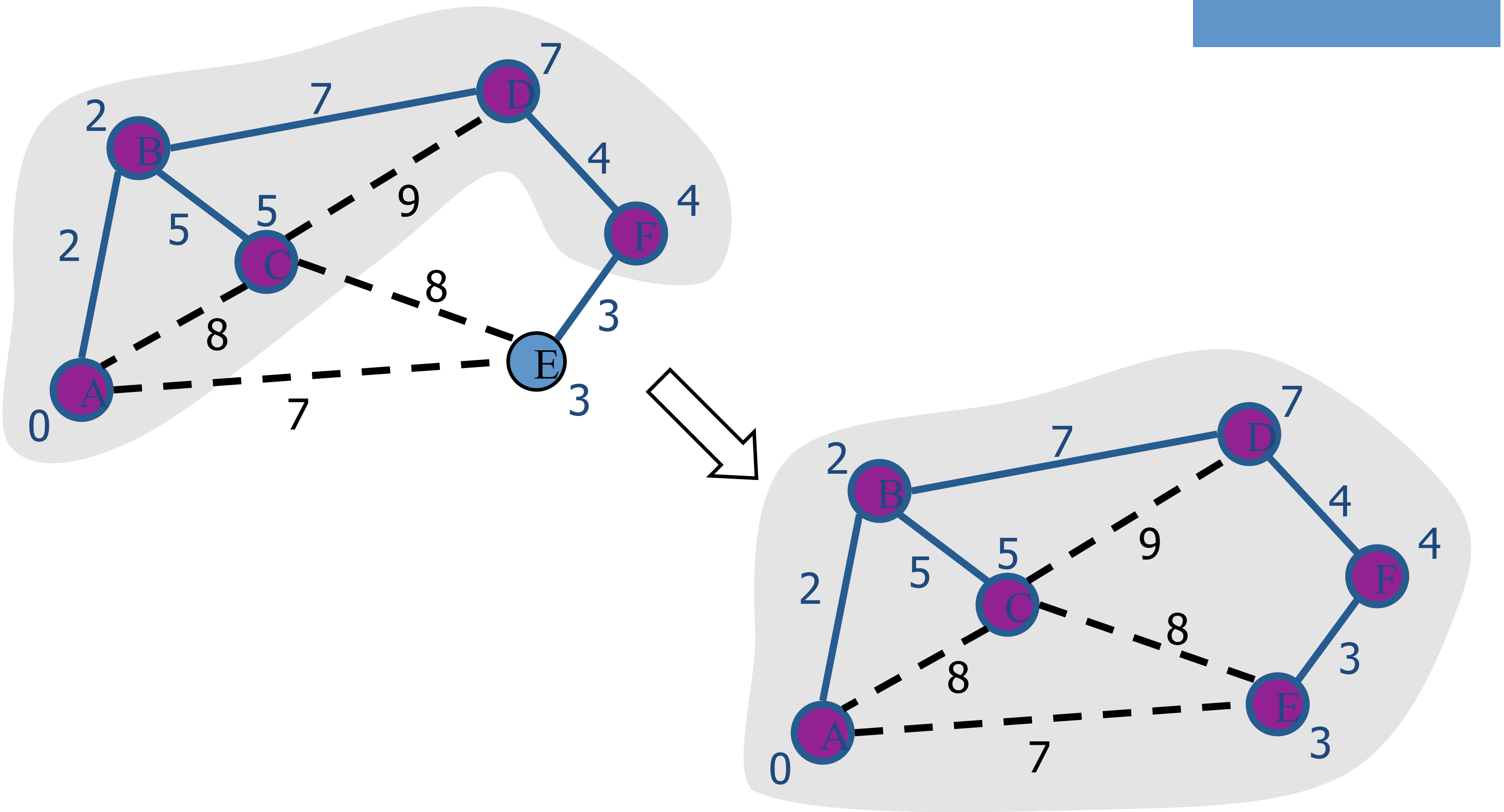
- A heap-based adaptable priority queue with location-aware entries stores the vertices outside the cloud
  - Key: distance
  - Value: vertex
  - Recall that method *replaceKey*(*l*,*k*) changes the key of entry *l*
- We store three labels with each vertex:
  - Distance
  - Parent edge in MST
  - Entry in priority queue

```
Algorithm PrimJarnikMST(G)
  Q ← new heap-based priority queue
  s ← a vertex of G
  for all v ∈ G.vertices()
    if v = s
      setDistance(v, 0)
    else
      setDistance(v, ∞)
    setParent(v, ∅)
    l ← Q.insert(getDistance(v), v)
    setLocator(v,l)
  while ¬Q.isEmpty()
    l ← Q.removeMin()
    u ← l.getValue()
    for all e ∈ G.incidentEdges(u)
      z ← G.opposite(u,e)
      r ← weight(e)
      if r < getDistance(z)
        setDistance(z, r)
        setParent(z,e)
        Q.replaceKey(getEntry(z), r)
```

Example



Example (contd.)



Baruvka's Algorithm

- Like Kruskal's Algorithm, Baruvka's algorithm grows many clusters at once and maintains a forest T
- Each iteration of the while loop halves the number of connected components in forest T
- The running time is $O(m \log n)$

```
Algorithm BaruvkaMST(G)
    T ← V  {just the vertices of G}
  while T has fewer than n − 1 edges do
    for each connected component C in T do
      Let edge e be the smallest-weight edge from C to another component in T
      if e is not already in T then
        Add edge e to T
  return T
```

Example of Baruvka's Algorithm (animated)