I. Abstract

This research derives the valuation formula for general cross-currency equity swaps. Under the HJM framework of international security economy, the valuation formula is derived by the use of domestic spot martingale measure. The pricing model is general in the sense that the settlement currency can be chosen to be any currencies. The study finds that the premium of exchange rate risk is important in the valuation process and this distinguishes it from domestic or cross-currency without exchange-rate risk swaps. Also, the hedging method of swaps is investigated.

II. Purposes of The Study

Equity swaps are important financial instruments in the over-the-counter markets. They provide an efficient way of international diversification and obtaining foreign equity returns without actual holding of foreign equities. Direct foreign equity investment may be tempered by some reasons, for example, tax consideration and the restriction of local government on the investment on equity of the foreigner.

There are relatively few papers investigate the pricing formula of international equity swaps. Rich (1995) decomposed an equity swaps into a portfolio of equity forward contracts and used a forward-start forward contract approach to value the basic equity swaps. Jarrow and Turnbull (1996) derived a formula for the fixed rate of a basic equity swap. Chance and Rich (1998) used arbitrage-free replicating portfolios to derive the valuation formulas for variety forms of equity swaps.

In this study, the valuation formula is derived for general equity swap in which the settlement currency can be set to be different to the underlying equity markets. Furthermore, the hedging strategies for the swaps are also investigated. The study sets up an international security economy under Heath, Jarrow, and Morton (1992) framework of stochastic interest rates. Then the dynamics are transformed to domestic spot martingale measure. The pricing formula can be derived in one of following three ways: domestic market method, foreign market method, and replication method. If we use the two former methods, we have to further transform the spot martingale measures to forward measure so as to
simplify the computation under stochastic interest rates. Here we use the replication method, since it also provides hedging strategies for cross-currency forward equity swap.

III. Results and Discussion

III.1. The International Security Economy

Let the underlying probability space be $(\Omega, F, P, (F_t^k)_{t=0}^\infty)$, where $(F_t^k)_{t=0}^\infty$ is the natural filtration generated by the $d$-dimensional standard Brownian motion $W = (W^1, \ldots, W^d)$. We specify the stochastic interest rates by the motion of forward rates:

\[df^i(t, T) = \alpha^i(t, T) dt + \sigma^i(t, T) dW^i,\]

where $i = h, k$ and $h$ for the domestic market and $k$ for the foreign market, respectively. Let the exchange rate of $k$ currency in units of domestic currency be denoted by $Q^k_t$ and its dynamics are

\[dQ^k_t = \alpha^k dt + \sigma^k dW^k.\]

Then the short term interest rates $r^i_t = f^i(t, t)$ and the prices of zero-coupon bonds are

\[B^i(t, T) = \exp(-\int_t^T f^i(u, u) du); \text{ the savings accounts are } B^i_t = \exp(\int_0^t f^i(u, u) du).\]

The dynamics of $B^i(t, T)$ under $P$ are

\[dB^i(t, T) = B^i(t, T)(a^i(t, T) dt + b^i(t, T) dW^i),\]

where

\[a^i(t, T) = \alpha^i_t - \alpha^i_t(t, T) + \frac{1}{2}[\sigma^i_t(t, T)], \quad b^i(t, T) = -\sigma^i_t(t, T), \text{ and } \alpha^i_t(t, u) = \int_0^u \alpha^i(t, u) du; \text{ } \sigma^i_t(t, u) = \int_0^u \sigma^i(t, u) du.\]

The other risky assets (called stocks) have dynamics as

\[dS^i_t = S^i_t(\mu^i dt + \xi^i dW^i).\]

If we use domestic savings account as the numeraire such that all relative prices in terms of it are martingales under a new probability measure $P^*$, which is called the domestic spot martingale measure. Then the Radon-Nikodym derivative is expressed as

\[\frac{dP^*}{dP} = \varepsilon_t(\eta \cdot W) = \exp\left(\int_0^T \eta_u \cdot dW^u - \frac{1}{2}\int_0^T |\eta_u|^2 du\right),\]

and $\eta$ satisfies the following equations:

\[-\frac{\sigma^i_t(t, T) + \frac{1}{2}|\sigma^i_t(t, T)|}{2} - \frac{\sigma^i_t(t, T) \eta_t}{2} = 0 \quad (5)\]

\[\mu^i_t - \frac{1}{2}\sigma^i_t(t, T) \eta_t = 0 \quad (6)\]

\[\mu^i_{\eta} + \frac{1}{2}\sigma^i_t(t, T) \eta_t = 0 \quad (7)\]

\[-r^i_t + (\sigma^i_t(t, T) + \frac{1}{2}|\sigma^i_t(t, T)|) \eta_t = 0 \quad (8)\]

\[\mu^i_t + \sigma^i_t \xi^i_t - r^i_t = 0 \quad (9)\]

Hence, under $P^*$, we have

\[d^h (t, T) = \sigma^h(t, T) \sigma^h(t, T) dt + \sigma^h(t, T) dW^h,\]

\[d^k (t, T) = \sigma^k(t, T) \sigma^k(t, T) dt + \sigma^k(t, T) dW^k.\]

\[dQ^h = Q^h(t^h - t^k) dt + \sigma^h dW^h,\]

\[dQ^k = Q^k(t^h - t^k) dt + \sigma^k dW^k.\]

\[dS^h = S^h(t_r^h dt + \xi^h dW^h),\]

\[dS^k = S^k((t_r^k - \sigma^k) dt + \xi^k dW^k).\]

where $W^r_t = W_t - \int_0^t \eta_u dW_u$ is a d-dimensional standard Brownian motion under $P^*$.

III. Derivation of The Formula

Applying the technique of measure changes, we can also derive the dynamics under foreign spot, domestic forward, and foreign forward martingale measures.

We now consider the pricing of cross-currency equity swaps. Suppose the swap begins at time $T_0 \geq t$, and the payment dates are $T_j$, $j = 1, 2, \ldots, n$. For simplicity, we assume $T_j - T_{j-1} = \delta$. For each period, the domestic investor receives $N_m R_m(T_{j-1}, T_j)$ and pays $N_m R_g(T_{j-1}, T_j)$, where $N_m$ is the notional principal denominated in currency $m$; $R_m$ and $R_g$ denote the equity returns of market $k$ and $g$ for the corresponding period, respectively. At time $t$, let its value be $CFES_t(k, g; m)$ for $N_m = 1$.

To derive $CFES_t(k, g; m)$, we may proceed by using domestic forward martingale measure. Also, we can derive it by replication approach. Since the two methods lead to the same valuation formula, we prefer here to present the method of replicating the payoff
of a cross-currency equity swap, due to this method also provides the way of hedging. The value of $C_{FE}(k, g; m)$ is

$$B_t^k E_p^j \left[ \sum_{j=1}^n \left( \frac{(R_k(T_{j-1}-T_j) - R_g(T_{j-1}-T_j))Q_{T_j}^m}{B_t^j} \right) \right]$$

$$= \sum_{j=1}^n B^k(t, T_j)E_{P_{T_j}} \left[ \frac{(R_k(T_{j-1}-T_j) - R_g(T_{j-1}-T_j))Q_{T_j}^m}{B_t^j} \right]$$

To derive $C_{FE}(k, g; m)$, we can replicate $(R_k(T_{j-1}-T_j) - R_g(T_{j-1}-T_j))Q_{T_j}^m$ under either $P^*$ or $P_t$, the resulting hedging strategies are identical.

First, consider a foreign-equity linked foreign exchange call option. Its payoff at time $T_j$ is

$$k(T_m T_j) (S_T - K)^+$$

For a put option:

$$k(T_m T_j) (K - S_T)^+$$

Now, at time $T_{j-1}$, we buy one home dollar stock $S_{T_{j-1}}^k$, buy $\frac{1}{(Q_{T_{j-1}}^m S_{T_{j-1}}^k)}$ units of put option and sell $\frac{1}{(Q_{T_{j-1}}^m S_{T_{j-1}}^k)}$ units of call option. At $T_j$, this portfolio has payoff $Q_{T_j}^m S_{T_j}^k$. The cost of the portfolio at $T_{j-1}$ is $1 + \left( \frac{P_m(T_{j-1}) - C_m(T_{j-1})}{(Q_{T_{j-1}}^m S_{T_{j-1}}^k)} \right)$. Do the same trading strategy for security $g$. We then discount this value to time $t$.

Furthermore, we can use domestic and foreign zero-coupon bonds to replicate $P_m(T_{j-1})$ and $C_m(T_{j-1})$, thus we can express the value of cross-currency equity swap at $t$ as $C_{FE}(k, g; m)$

$$= \sum_{j=1}^n Q_{T_j}^m B^k(t, T_j) \{ (1 + f(t, T_{j-1}, T_j))H_m(t, T_{j-1}, T_j)$$

$$- (1 + f(t, T_{j-1}, T_j))H^m_m(t, T_{j-1}, T_j) \}, \quad (15)$$

where $f^j(t, T_{j-1}, T_j)$ is the forward rate at $t$ for the period $[T_{j-1}, T_j]$, and $H^m_m(t, T_{j-1}, T_j)$

$$= \exp \left\{ \int_{T_{j-1}}^{T_j} \left[ \begin{array}{c} \xi_u^m \sigma_u^m - \sigma_u^m + b_i(u, T_j) \\ - \xi_u^m b_i(u, T_j)du \end{array} \right] \right\}$$

IV. Evaluation of The Study

This study derives the valuation formula for the fair prices of cross-currency forward equity swaps. In this general setting, various forms of equity swaps can be applied by substituting proper conditions into (15). Furthermore, using the replication approach provides a practical way of hedging these financial instruments.

For future study, we may conduct numerical simulation. Also, we may derive the option prices on these swaps and consider the factor of default risks. After these works have been done, the models can be used efficiently for the financial industry.

V. Reference


