Review the Concepts

3.1 Explain the difference between the sample average $\bar{Y}$ and the population mean.

3.2 Explain the difference between an estimator and an estimate. Provide an example of each.

3.3 A population distribution has a mean of 10 and a variance of 16. Determine the mean and variance of $\bar{Y}$ from an i.i.d. sample from this population for (a) $n = 10$; (b) $n = 100$; and (c) $n = 1000$. Relate your answers to the law of large numbers.

3.4 What role does the central limit theorem play in statistical hypothesis testing? In the construction of confidence intervals?

3.5 What is the difference between a null and alternative hypothesis? Among size, significance level, and power? Between a one-sided and two-sided alternative hypothesis?

3.6 Why does a confidence interval contain more information than the result of a single hypothesis test?

3.7 Explain why the differences-of-means estimator, applied to data from a randomized controlled experiment, is an estimator of the treatment effect.

3.8 Sketch a hypothetical scatterplot for a sample of size 10 for two random variables with a population correlation of (a) 1.0; (b) −1.0; (c) 0.9; (d) −0.5; (e) 0.0.

Exercises

$\sqrt{3.1}$ In a population $\mu_Y = 100$ and $\sigma_Y^2 = 43$. Use the central limit theorem to answer the following questions:

a. In a random sample of size $n = 100$, find $\Pr(\bar{Y} < 101)$.

b. In a random sample of size $n = 64$, find $\Pr(101 < \bar{Y} < 103)$.

c. In a random sample of size $n = 165$, find $\Pr(\bar{Y} > 98)$.

3.2 Let $Y$ be a Bernoulli random variable with success probability $\Pr(Y = 1) = p$, and let $Y_1, \ldots , Y_n$ be i.i.d. draws from this distribution. Let $\hat{p}$ be the fraction of successes (1s) in this sample.

a. Show that $\hat{p} = \bar{Y}$.

b. Show that $\hat{p}$ is an unbiased estimator of $p$.

c. Show that $\text{var}(\hat{p}) = p(1 - p)/n$. 
3.3 In a survey of 400 likely voters, 215 responded that they would vote for the incumbent and 185 responded that they would vote for the challenger. Let \( p \) denote the fraction of all likely voters who preferred the incumbent at the time of the survey, and let \( \hat{p} \) be the fraction of survey respondents who preferred the incumbent.

a. Use the survey results to estimate \( p \).

b. Use the estimator of the variance of \( \hat{p}, \hat{p}(1 - \hat{p})/n \), to calculate the standard error of your estimator.

c. What is the \( p \)-value for the test \( H_0: p = 0.5 \) vs. \( H_1: p \neq 0.5 \)?

d. What is the \( p \)-value for the test \( H_0: p = 0.5 \) vs. \( H_1: p > 0.5 \)?

e. Why do the results from (c) and (d) differ?

f. Did the survey contain statistically significant evidence that the incumbent was ahead of the challenger at the time of the survey? Explain.

3.4 Using the data in Exercise 3.3:

a. Construct a 95% confidence interval for \( p \).

b. Construct a 99% confidence interval for \( p \).

c. Why is the interval in (b) wider than the interval in (a)?

d. Without doing any additional calculations, test the hypothesis \( H_0: p = 0.50 \) vs. \( H_1: p \neq 0.50 \) at the 5% significance level.

3.5 A survey of 1055 registered voters is conducted, and the voters are asked to choose between candidate A and candidate B. Let \( p \) denote the fraction of voters in the population who prefer candidate A, and let \( \hat{p} \) denote the fraction of voters in the sample who prefer Candidate A.

a. You are interested in the competing hypotheses: \( H_0: p = 0.5 \) vs. \( H_1: p \neq 0.5 \). Suppose that you decide to reject \( H_0 \) if \( |\hat{p} - 0.5| > 0.02 \).

i. What is the size of this test?

ii. Compute the power of this test if \( p = 0.53 \).

b. In the survey \( \hat{p} = 0.54 \).

i. Test \( H_0: p = 0.5 \) vs. \( H_1: p \neq 0.5 \) using a 5% significance level.

ii. Test \( H_0: p = 0.5 \) vs. \( H_1: p > 0.5 \) using a 5% significance level.

iii. Construct a 95% confidence interval for \( p \).

iv. Construct a 99% confidence interval for \( p \).

v. Construct a 50% confidence interval for \( p \).
c. Suppose that the survey is carried out 20 times, using independently selected voters in each survey. For each of these 20 surveys, a 95\% confidence interval for \( p \) is constructed.

i. What is the probability that the true value of \( p \) is contained in all 20 of these confidence intervals?

ii. How many of these confidence intervals do you expect to contain the true value of \( p \)?

d. In survey jargon, the "margin of error" is \( 1.96 \times \text{SE}(\hat{p}) \); that is, it is \( \frac{1}{2} \) times the length of 95\% confidence interval. Suppose you wanted to design a survey that had a margin of error of at most 1\%. That is, you wanted \( \Pr(|\hat{p} - p| > 0.01) \leq 0.05 \). How large should \( n \) be if the survey uses simple random sampling?

3.6 Let \( Y_1, \ldots, Y_n \) be i.i.d. draws from a distribution with mean \( \mu \). A test of \( H_0: \mu = 5 \) versus \( H_1: \mu \neq 5 \) using the usual \( t \)-statistic yields a \( p \)-value of 0.03.

a. Does the 95\% confidence interval contain \( \mu = 5 \)? Explain.

b. Can you determine if \( \mu = 6 \) is contained in the 95\% confidence interval? Explain.

3.7 In a given population, 11\% of the likely voters are African American. A survey using a simple random sample of 600 land-line telephone numbers finds 8\% African Americans. Is there evidence that the survey is biased? Explain.

3.8 A new version of the SAT test is given to 1000 randomly selected high school seniors. The sample mean test score is 1110 and the sample standard deviation is 123. Construct a 95\% confidence interval for the population mean test score for high school seniors.

3.9 Suppose that a lightbulb manufacturing plant produces bulbs with a mean life of 2000 hours and a standard deviation of 200 hours. An inventor claims to have developed an improved process that produces bulbs with a longer mean life and the same standard deviation. The plant manager randomly selects 100 bulbs produced by the process. She says that she will believe the inventor's claim if the sample mean life of the bulbs is greater than 2100 hours; otherwise, she will conclude that the new process is no better than the old process. Let \( \mu \) denote the mean of the new process. Consider the null and alternative hypothesis \( H_0: \mu = 2000 \) vs. \( H_1: \mu > 2000 \).

a. What is the size of the plant manager's testing procedure?

b. Suppose that the new process is in fact better and has a mean bulb life of 2150 hours. What is the power of the plant manager's testing procedure?
c. What testing procedure should the plant manager use if she wants the size of her test to be 5%?

3.10 Suppose a new standardized test is given to 100 randomly selected third-grade students in New Jersey. The sample average score $\bar{Y}$ on the test is 58 points and the sample standard deviation, $s_Y$, is 8 points.

a. The authors plan to administer the test to all third-grade students in New Jersey. Construct a 95% confidence interval for the mean score of all New Jersey third graders.

b. Suppose the same test is given to 200 randomly selected third graders from Iowa, producing a sample average of 62 points and sample standard deviation of 11 points. Construct a 90% confidence interval for the difference in mean scores between Iowa and New Jersey.

c. Can you conclude with a high degree of confidence that the population means for Iowa and New Jersey students are different? (What is the standard error of the difference in the two sample means? What is the p-value of the test of no difference in means versus some difference?)

3.11 Consider the estimator $\bar{Y}$, defined in Equation (3.1). Show that (a) $E(\bar{Y}) = \mu_Y$ and (b) $\text{var}(\bar{Y}) = 1.25\sigma_Y^2/n$.

3.12 To investigate possible gender discrimination in a firm, a sample of 100 men and 64 women with similar job descriptions are selected at random. A summary of the resulting monthly salaries follows:

<table>
<thead>
<tr>
<th></th>
<th>Average Salary ($\bar{Y}$)</th>
<th>Standard Deviation ($s_Y$)</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Men</td>
<td>$3100$</td>
<td>$200$</td>
<td>100</td>
</tr>
<tr>
<td>Women</td>
<td>$2900$</td>
<td>$320$</td>
<td>64</td>
</tr>
</tbody>
</table>

a. What do these data suggest about wage differences in the firm? Do they represent statistically significant evidence that wages of men and women are different? (To answer this question, first state the null and alternative hypothesis; second, compute the relevant $t$-statistic; third, compute the $p$-value associated with the $t$-statistic; and finally use the $p$-value to answer the question.)

b. Do these data suggest that the firm is guilty of gender discrimination in its compensation policies? Explain.

3.13 Data on fifth-grade test scores (reading and mathematics) for 420 school districts in California yield $\bar{Y} = 646.2$ and standard deviation $s_Y = 19.5$,
a. Construct a 95% confidence interval for the mean test score in the population.

b. When the districts were divided into districts with small classes (≤ 20 students per teacher) and large classes (≥ 20 students per teacher), the following results were found:

<table>
<thead>
<tr>
<th>Class Size</th>
<th>Average Score ((\bar{Y}))</th>
<th>Standard Deviation ((s_Y))</th>
<th>(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td>657.4</td>
<td>19.4</td>
<td>238</td>
</tr>
<tr>
<td>Large</td>
<td>650.0</td>
<td>17.9</td>
<td>182</td>
</tr>
</tbody>
</table>

Is there statistically significant evidence that the districts with smaller classes have higher average test scores? Explain.

3.14 Values of height in inches (\(X\)) and weight in pounds (\(Y\)) are recorded from a sample of 300 male college students. The resulting summary statistics are \(\bar{X} = 70.5\) inches; \(\bar{Y} = 158\) lbs; \(s_X = 1.8\) inches; \(s_Y = 14.2\) lbs; \(s_{XY} = 21.73\) inches \(\times\) lbs, and \(r_{XY} = 0.85\). Convert these statistics to the metric system (meters and kilograms).

3.15 The CNN/USA Today/Gallup poll conducted on September 3–5, 2004, surveyed 755 likely voters; 405 reported a preference for President George W. Bush, and 350 reported a preference for Senator John Kerry. The CNN/USA Today/Gallup poll conducted on October 1–3, 2004, surveyed 756 likely voters; 378 reported a preference for Bush, and 378 reported a preference for Kerry.

a. Construct a 95% confidence interval for the fraction of likely voters in the population who favored Bush in early September 2004.

b. Construct a 95% confidence interval for the fraction of likely voters in the population who favored Bush in early October 2004.

c. Was there a statistically significant change in voters' opinions across the two dates?

3.16 Grades on a standardized test are known to have a mean of 1000 for students in the United States. The test is administered to 453 randomly selected students in Florida; in this sample, the mean is 1013 and the standard deviation (\(s\)) is 108.

a. Construct a 95% confidence interval for the average test score for Florida students.
b. Use Equation (2.33) to show that \( \text{cov}(\overline{Y}, Y_i) = \sigma_Y^2 / n \).

c. Use the results in parts (a) and (b) to show that \( E(s_Y^2) = \sigma_Y^2 \).

\[ 3.19 \]

a. \( \overline{Y} \) is an unbiased estimator of \( \mu_Y \). Is \( \overline{Y}^2 \) an unbiased estimator of \( \mu_Y^2 \)?

b. \( \overline{Y} \) is a consistent estimator of \( \mu_Y \). Is \( \overline{Y}^2 \) a consistent estimator of \( \mu_Y^2 \)?

\[ 3.20 \]

Suppose that \((X_i, Y_i)\) are i.i.d. with finite fourth moments. Prove that the sample covariance is a consistent estimator of the population covariance, that is, \( s_{XY} \xrightarrow{p} \sigma_{XY} \), where \( s_{XY} \) is defined in Equation (3.24). (Hint: Use the strategy of Appendix 3.3 and the Cauchy-Schwarz inequality.)

\[ 3.21 \]

Show that the pooled standard error \([SE_{pooled}(\overline{Y}_m - \overline{Y}_n)]\) given following Equation (3.23) equals the usual standard error for the difference in means in Equation (3.19) when the two group sizes are the same \((n_m = n_n)\).

**Empirical Exercise**

**E3.1** On the text Web site [www.aw-bc.com/stock_watson](http://www.aw-bc.com/stock_watson) you will find a data file **CPS92_04** that contains an extended version of the dataset used in Table 3.1 of the text for the years 1992 and 2004. It contains data on full-time, full-year workers, age 25–34, with a high school diploma or B.A./B.S. as their highest degree. A detailed description is given in **CPS92_04_Description**, available on the Web site. Use these data to answer the following questions.


b. In 2004, the value of the Consumer Price Index (CPI) was 188.9. In 1992, the value of the CPI was 140.3. Repeat (a) but use AHE measured in real 2004 dollars ($2004); that is, adjust the 1992 data for the price inflation that occurred between 1992 and 2004.

c. If you were interested in the change in workers' purchasing power from 1992 to 2004, would you use the results from (a) or from (b)? Explain.

d. Use the 2004 data to construct a 95% confidence interval for the mean of AHE for high school graduates. Construct a 95% confidence