Structural Induction on Trees

Structural Induction on Trees

Structural induction is not limited to lists; it applies to any tree structure.

The general induction principle is the following:

To prove a property $P(t)$ for all trees t of a certain type,

- show that $P(l)$ holds for all leaves l of a tree,
- for each type of internal node t with subtrees $s_1, \ldots, s_n$, show that

$$P(s_1) \land \ldots \land P(s_n) \text{ implies } P(t).$$

Example: IntSets

Recall our definition of `IntSet` with the operations `contains` and `incl`:

```
abstract class IntSet {
   def incl(x: Int): IntSet
   def contains(x: Int): Boolean
}

object Empty extends IntSet {
  def contains(x: Int): Boolean = false
  def incl(x: Int): IntSet = NonEmpty(x, Empty, Empty)
}
```

Example: IntSets (2)

```
case class NonEmpty(elem: Int, left: IntSet, right: IntSet) extends IntSet {

  def contains(x: Int): Boolean =
    if (x < elem) left contains x
    else if (x > elem) right contains x
    else true

  def incl(x: Int): IntSet =
    if (x < elem) NonEmpty(elem, left incl x, right)
    else if (x > elem) NonEmpty(elem, left, right incl x)
    else this
}
```

The Laws of IntSet

What does it mean to prove the correctness of this implementation?

One way to define and show the correctness of an implementation consists of proving the laws that it respects.

In the case of `IntSet`, we have the following three laws:

For any set `s`, and elements `x` and `y`:

```
Empty contains x          =  false
(s incl x) contains x     =  true
(s incl x) contains y     =  s contains y        if x != y
```

(In fact, we can show that these laws completely characterize the desired data type).

Proving the Laws of IntSet (1)

How can we prove these laws?

Proposition 1: `Empty contains x = false`.

Proof: According to the definition of `contains` in `Empty`.

Proving the Laws of IntSet (2)

Proposition 2: `(s incl x) contains x = true`

Proof by structural induction on `s`.

Base case: `Empty`

```
(Empty incl x) contains x
```

Proving the Laws of IntSet (2)

Proposition 2: `(s incl x) contains x = true`

Proof by structural induction on `s`.

Base case: `Empty`

```
(Empty incl x) contains x

=   NonEmpty(x, Empty, Empty) contains x    // by definition of Empty.incl
```

Proving the Laws of IntSet (2)

Proposition 2: `(s incl x) contains x = true`

Proof by structural induction on `s`.

Base case: `Empty`

```
(Empty incl x) contains x

=   NonEmpty(x, Empty, Empty) contains x   // by definition of Empty.incl

=   true                                   // by definition of NonEmpty.contains
```

Proving the Laws of IntSet (3)

Induction step: `NonEmpty(x, l, r)`

```
(NonEmpty(x, l, r) incl x) contains x
```

NonEmpty(z, l, r)

z = x
z ≠ x

Proving the Laws of IntSet (3)

Induction step: `NonEmpty(x, l, r)`

```
(NonEmpty(x, l, r) incl x) contains x

=   NonEmpty(x, l, r) contains x          //  by definition of NonEmpty.incl
```

Proving the Laws of IntSet (3)

Induction step: `NonEmpty(x, l, r)`

```
(NonEmpty(x, l, r) incl x) contains x

=   NonEmpty(x, l, r) contains x          //  by definition of NonEmpty.incl

=   true                                  //  by definition of NonEmpty.contains
```

Proving the Laws of IntSet (4)

Induction step: `NonEmpty(y, l, r)` **where** `y < x`

```
(NonEmpty(y, l, r) incl x) contains x
```

Proving the Laws of IntSet (4)

Induction step: `NonEmpty(y, l, r)` **where** `y < x`

```
(NonEmpty(y, l, r) incl x) contains x

=   NonEmpty(y, l, r incl x) contains x   // by definition of NonEmpty.incl
```

Proving the Laws of IntSet (4)

Induction step: `NonEmpty(y, l, r)` **where** `y < x`

```
(NonEmpty(y, l, r) incl x) contains x

=   NonEmpty(y, l, r incl x) contains x   // by definition of NonEmpty.incl

=   (r incl x) contains x                 // by definition of NonEmpty.contains
```

Proving the Laws of IntSet (4)

Induction step: `NonEmpty(y, l, r)` **where** `y < x`

```
(NonEmpty(y, l, r) incl x) contains x

=   NonEmpty(y, l, r incl x) contains x   // by definition of NonEmpty.incl

=   (r incl x) contains x                 // by definition of NonEmpty.contains

=   true                                  // by the induction hypothesis
```

Proving the Laws of IntSet (4)

Induction step: `NonEmpty(y, l, r)` **where** `y < x`

```
(NonEmpty(y, l, r) incl x) contains x

=   NonEmpty(y, l, r incl x) contains x    // by definition of NonEmpty.incl

=   (r incl x) contains x                  // by definition of NonEmpty.contains

=   true                                   // by the induction hypothesis
```

Induction step: `NonEmpty(y, l, r)` **where** `y > x` is analogous

Proving the Laws of IntSet (5)

Proposition 3: If `x != y` then

```
(xs incl y) contains x  =  xs contains x.
```

Proof by structural induction on `s`. Assume that `y < x` (the dual case `x < y` is analogous).

Base case: `Empty`

```
(Empty incl y) contains x                    // to show: =  Empty contains x
```

Proving the Laws of IntSet (5)

Proposition 3: If `x != y` then

```
(xs incl y) contains x  =  xs contains x.
```

Proof by structural induction on `s`. Assume that `y < x` (the dual case `x < y` is analogous).

Base case: `Empty`

```
  (Empty incl y) contains x                // to show: =  Empty contains x

=   NonEmpty(y, Empty, Empty) contains x   // by definition of Empty.incl
```

Proving the Laws of IntSet (5)

Proposition 3: If `x != y` then

```
(xs incl y) contains x  =  xs contains x.
```

Proof by structural induction on `s`. Assume that `y < x` (the dual case `x < y` is analogous).

Base case: `Empty`

```
  (Empty incl y) contains x                 // to show: =  Empty contains x

=   NonEmpty(y, Empty, Empty) contains x    // by definition of Empty.incl

=   Empty contains x                        // by definition of NonEmpty.contains
```

Proving the Laws of IntSet (6)

For the inductive step, we need to consider a tree `NonEmpty(z, l, r)`. We distinguish five cases:

1. `z = x`
2. `z = y`
3. `z < y < x`
4. `y < z < x`
5. `y < x < z`

First Two Cases: `z = x, z = y`

Induction step: `NonEmpty(x, l, r)`

```
(NonEmpty(x, l, r) incl y) contains x  // to show: =  NonEmpty(x, l, r) contains x
```

First Two Cases: z = x, z = y

Induction step: `NonEmpty(x, l, r)`

```
(NonEmpty(x, l, r) incl y) contains x  // to show: =  NonEmpty(x, l, r) contains x

=   NonEmpty(x, l incl y, r) contains x    // by definition of NonEmpty.incl
```

First Two Cases: `z = x, z = y`

Induction step: `NonEmpty(x, l, r)`

```
(NonEmpty(x, l, r) incl y) contains x  // to show: =  NonEmpty(x, l, r) contains x

=   NonEmpty(x, l incl y, r) contains x    // by definition of NonEmpty.incl

=   true                                   // by definition of NonEmpty.contains
```

First Two Cases: `z = x`, `z = y`

Induction step: `NonEmpty(x, l, r)`

```
(NonEmpty(x, l, r) incl y) contains x  // to show: =  NonEmpty(x, l, r) contains x

=   NonEmpty(x, l incl y, r) contains x    // by definition of NonEmpty.incl

=   true                                   // by definition of NonEmpty.contains

=   NonEmpty(x, l, r) contains x           // by definition of NonEmpty.contains
```

First Two Cases: `z = x, z = y`

Induction step: `NonEmpty(x, l, r)`

```
(NonEmpty(x, l, r) incl y) contains x  // to show: =  NonEmpty(x, l, r) contains x

=   NonEmpty(x, l incl y, r) contains x    // by definition of NonEmpty.incl

=   true                                   // by definition of NonEmpty.contains

=   NonEmpty(x, l, r) contains x           // by definition of NonEmpty.contains
```

Induction step: `NonEmpty(y, l, r)`

```
(NonEmpty(y, l, r) incl y) contains x  // to show: =  NonEmpty(y, l, r) contains x
```

First Two Cases: z = x, z = y

Induction step: NonEmpty(x, l, r)

```
 (NonEmpty(x, l, r) incl y) contains x  // to show: =  NonEmpty(x, l, r) contains x

 =   NonEmpty(x, l incl y, r) contains x    // by definition of NonEmpty.incl

 =   true                                   // by definition of NonEmpty.contains

 =   NonEmpty(x, l, r) contains x           // by definition of NonEmpty.contains
```

Induction step: NonEmpty(y, l, r)

```
 (NonEmpty(y, l, r) incl y) contains x  // to show: =  NonEmpty(y, l, r) contains x

 =    NonEmpty(y, l, r) contains x          // by definition of NonEmpty.incl
```

Case `z < y`

Induction step: `NonEmpty(z, l, r)` **where** `z < y < x`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x
```

Case z < y

Induction step: `NonEmpty(z, l, r)` **where** `z < y < x`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x

=   NonEmpty(z, l, r incl y) contains x    // by definition of NonEmpty.incl
```

Case z < y

Induction step: `NonEmpty(z, l, r)` **where** `z < y < x`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x

=   NonEmpty(z, l, r incl y) contains x    // by definition of NonEmpty.incl

=   (r incl y) contains x                  // by definition of NonEmpty.contains
```

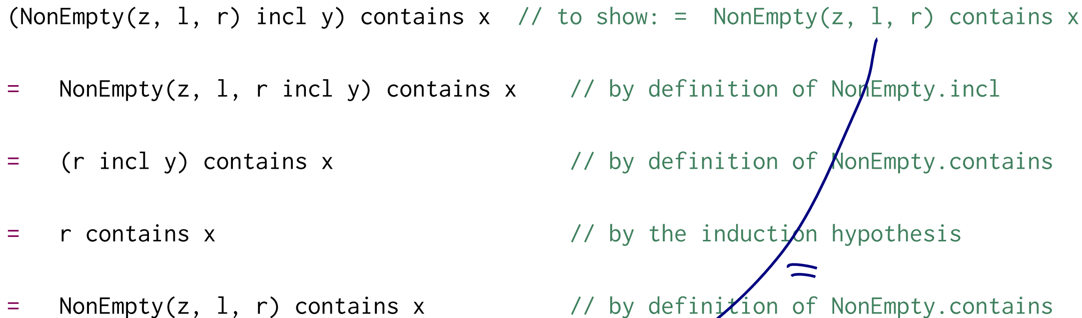
Case z < y

Induction step: `NonEmpty(z, l, r)` **where** `z < y < x`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x

=   NonEmpty(z, l, r incl y) contains x    // by definition of NonEmpty.incl

=   (r incl y) contains x                  // by definition of NonEmpty.contains

=   r contains x                           // by the induction hypothesis
```

Case z < y

Induction step: `NonEmpty(z, l, r)` **where** `z < y < x`



Case y < z < x

Induction step: `NonEmpty(z, l, r)` **where** `y < z < x`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x
```

Case y < z < x

Induction step: `NonEmpty(z, l, r)` **where** `y < z < x`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x

=   NonEmpty(z, l incl y, r) contains x    // by definition of NonEmpty.incl
```

Case y < z < x

Induction step: `NonEmpty(z, l, r)` **where** `y < z < x`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x

=   NonEmpty(z, l incl y, r) contains x    // by definition of NonEmpty.incl

=   r contains x                           // by definition of NonEmpty.contains
```

Case y < z < x

Induction step: `NonEmpty(z, l, r)` **where** `y < z < x`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x

=   NonEmpty(z, l incl y, r) contains x    // by definition of NonEmpty.incl

=   r contains x                           // by definition of NonEmpty.contains

=   NonEmpty(z, l, r) contains x           // by definition of NonEmpty.contains
```

Case `x < z`

Induction step: `NonEmpty(z, l, r)` **where** `y < x < z`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x
```

Case x < z

Induction step: `NonEmpty(z, l, r)` **where** `y < x < z`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x

=   NonEmpty(z, l incl y, r) contains x    // by definition of NonEmpty.incl
```

Case x < z

Induction step: `NonEmpty(z, l, r)` **where** `y < x < z`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x

=   NonEmpty(z, l incl y, r) contains x    // by definition of NonEmpty.incl

=   (l incl y) contains x                  // by definition of NonEmpty.contains
```

Case x < z

Induction step: `NonEmpty(z, l, r)` **where** `y < x < z`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x

=   NonEmpty(z, l incl y, r) contains x     // by definition of NonEmpty.incl

=   (l incl y) contains x                   // by definition of NonEmpty.contains

=   l contains x                            // by the induction hypothesis
```

Case `x < z`

Induction step: `NonEmpty(z, l, r)` **where** `y < x < z`

```
(NonEmpty(z, l, r) incl y) contains x  // to show: =  NonEmpty(z, l, r) contains x

=   NonEmpty(z, l incl y, r) contains x    // by definition of NonEmpty.incl

=   (l incl y) contains x                  // by definition of NonEmpty.contains

=   l contains x                           // by the induction hypothesis

=   NonEmpty(z, l, r) contains x           // by definition of NonEmpty.contains
```

These are all the cases, so the proposition is established.

Exercise (Hard)

Suppose we add a function `union` to `IntSet`:

```
abstract class IntSet { ...
  def union(other: IntSet): IntSet
}
object Empty extends IntSet { ...
  def union(other: IntSet) = other
}
class NonEmpty(x: Int, l: IntSet, r: IntSet) extends IntSet { ...
  def union(other: IntSet): IntSet = (l union (r union (other))) incl x
}
```

Exercise (Hard)

The correctness of `union` can be translated into the following law:

Proposition 4:

```
(xs union ys) contains x  =  xs contains x || ys contains x
```

Show proposition 4 by using structural induction on `xs`.

Streams

Collections and Combinatorial Search

We've seen a number of immutable collections that provide powerful operations, in particular for combinatorial search.

For instance, to find the second prime number between 1000 and 10000:

```
((1000 to 10000) filter isPrime)(1)
```

This is *much* shorter than the recursive alternative:

```
def secondPrime(from: Int, to: Int) = nthPrime(from, to, 2)
def nthPrime(from: Int, to: Int, n: Int): Int =
  if (from >= to) throw new Error(”no prime”)
  else if (isPrime(from))
    if (n == 1) from else nthPrime(from + 1, to, n - 1)
  else nthPrime(from + 1, to, n)
```

Performance Problem

But from a standpoint of performance,

```
((1000 to 10000) filter isPrime)(1)
```

is pretty bad; it constructs *all* prime numbers between `1000` and `10000` in a list, but only ever looks at the first two elements of that list.

Reducing the upper bound would speed things up, but risks that we miss the second prime number all together.

Delayed Evaluation

However, we can make the short-code efficient by using a trick:

> *Avoid computing the tail of a sequence until it is needed for the evaluation result (which might be never)*

This idea is implemented in a new class, the `Stream`.

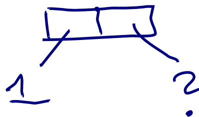
Streams are similar to lists, but their tail is evaluated only *on demand*.

Defining Streams

Streams are defined from a constant `Stream.empty` and a constructor `Stream.cons`.

For instance,

```
val xs = Stream.cons(1, Stream.cons(2, Stream.empty))
```



They can also be defined like the other collections by using the object `Stream` as a factory.

```
Stream(1, 2, 3)
```

The `toStream` method on a collection will turn the collection into a stream:

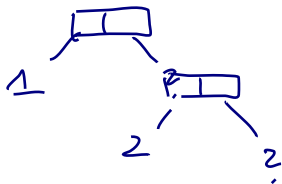
```
(1 to 1000).toStream
```

> *res0: Stream[Int] = Stream(1, ?)*

Stream Ranges

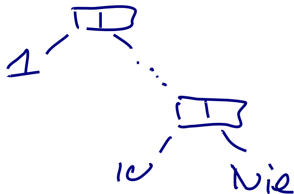
Let's try to write a function that returns (lo until hi).toStream directly:

```
def streamRange(lo: Int, hi: Int): Stream[Int] =
  if (lo >= hi) Stream.empty
  else Stream.cons(lo, streamRange(lo + 1, hi))
```



Compare to the same function that produces a list:

```
def listRange(lo: Int, hi: Int): List[Int] =
  if (lo >= hi) Nil
  else lo :: listRange(lo + 1, hi)
```



Comparing the Two Range Functions

The functions have almost identical structure yet they evaluate quite differently.

- `listRange(start, end)` will produce a list with `end - start` elements and return it.
- `streamRange(start, end)` returns a single object of type `Stream` with `start` as head element.
- The other elements are only computed when they are needed, where "needed" means that someone calls `tail` on the stream.

Methods on Streams

`Stream` supports almost all methods of `List`.

For instance, to find the second prime number between 1000 and 10000:

```
((1000 to 10000).toStream filter isPrime)(1)
```

Stream Cons Operator

The one major exception is `::`.

`x :: xs` always produces a list, never a stream.

There is however an alternative operator `#::` which produces a stream.

```
x #:: xs  ==   Stream.cons(x, xs)
```

`#::` can be used in expressions as well as patterns.

Implementation of Streams

The implementation of streams is quite close to the one of lists.

Here's the trait `Stream`:

```
trait Stream[+A] extends Seq[A] {
  def isEmpty: Boolean
  def head: A
  def tail: Stream[A]
  ...
}
```

As for lists, all other methods can be defined in terms of these three.

Implementation of Streams(2)

Concrete implementations of streams are defined in the `Stream` companion object. Here's a first draft:

```
object Stream {
  def cons[T](hd: T, tl: => Stream[T]) = new Stream[T] {
    def isEmpty = false
    def head = hd
    def tail = tl
  }
  val empty = new Stream[Nothing] {
    def isEmpty = true
    def head = throw new NoSuchElementException("empty.head")
    def tail = throw new NoSuchElementException("empty.tail")
  }
}
```

Difference to List

The only important difference between the implementations of `List` and `Stream` concern `tl`, the second parameter of `Stream.cons`.

For streams, this is a by-name parameter.

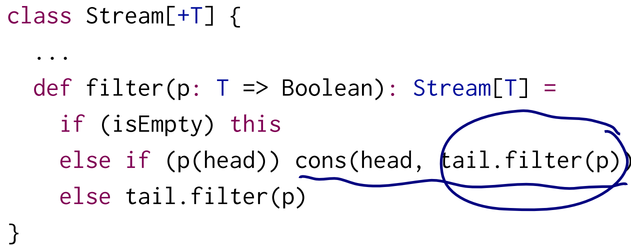
That's why the second argument to `Stream.cons` is not evaluated at the point of call.

Instead, it will be evaluated each time someone calls `tail` on a `Stream` object.

Other Stream Methods

The other stream methods are implemented analogously to their list counterparts.

For instance, here's `filter`:



```
class Stream[+T] {
  ...
  def filter(p: T => Boolean): Stream[T] =
    if (isEmpty) this
    else if (p(head)) cons(head, tail.filter(p))
    else tail.filter(p)
}
```

Exercise

Consider this modification of `streamRange`.

```
def streamRange(lo: Int, hi: Int): Stream[Int] = {
  print(lo+" ")
  if (lo >= hi) Stream.empty
  else Stream.cons(lo, streamRange(lo + 1, hi))
}
```

When you write `streamRange(1, 10).take(3).toList` what gets printed?

O Nothing

O 1

O 1 2 3

O 1 2 3 4

O 1 2 3 4 5 6 7 8 9

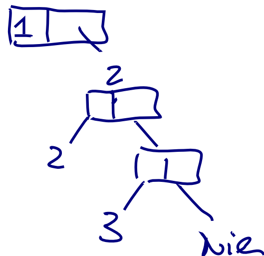
Exercise

Consider this modification of `streamRange`.

```
def streamRange(lo: Int, hi: Int): Stream[Int] = {
  print(lo+" ")
  if (lo >= hi) Stream.empty
  else Stream.cons(lo, streamRange(lo + 1, hi))
}
```

When you write `streamRange(1, 10).take(3).toList` what gets printed?

O	Nothing
O	1
●	1 2 3
O	1 2 3 4
O	1 2 3 4 5 6 7 8 9



Lazy Evaluation

Lazy Evaluation

The proposed implementation suffers from a serious potential performance problem: If `tail` is called several times, the corresponding stream will be recomputed each time.

This problem can be avoided by storing the result of the first evaluation of `tail` and re-using the stored result instead of recomputing `tail`.

This optimization is sound, since in a purely functional language an expression produces the same result each time it is evaluated.

We call this scheme *lazy evaluation* (as opposed to *by-name evaluation* in the case where everything is recomputed, and *strict evaluation* for normal parameters and `val` definitions.)

Lazy Evaluation in Scala

Haskell is a functional programming language that uses lazy evaluation by default.

Scala uses strict evaluation by default, but allows lazy evaluation of value definitions with the `lazy val` form:

```
lazy val x = expr
```

def x = expr

Exercise:

Consider the following program:

```
def expr = {
  val x = { print("x"); 1 }
  lazy val y = { print("y"); 2 }
  def z = { print("z"); 3 }
  z + y + x + z + y + x
}
expr
```

xzyz

If you run this program, what gets printed as a side effect of evaluating `expr`?

- O zyxzyx
- O xyzz
- O something else
- ● xzyz
- O zyzz

Lazy Vals and Streams

Using a lazy value for `tail`, `Stream.cons` can be implemented more efficiently:

```
def cons[T](hd: T, tl: => Stream[T]) = new Stream[T] {
  def head = hd
  lazy val tail = tl
  ...
}
```

Seeing it in Action

To convince ourselves that the implementation of streams really does avoid unnecessary computation, let's observe the execution trace of the expression:

```
(streamRange(1000, 10000) filter isPrime) apply 1
```

Seeing it in Action

To convince ourselves that the implementation of streams really does avoid unnecessary computation, let's observe the execution trace of the expression:

```
      (streamRange(1000, 10000) filter isPrime) apply 1

 -->  (if (1000 >= 10000) empty                // by expanding streamRange
       else cons(1000, streamRange(1000 + 1, 10000))
      .filter(isPrime).apply(1)
```

Seeing it in Action

To convince ourselves that the implementation of streams really does avoid unnecessary computation, let's observe the execution trace of the expression:

```
      (streamRange(1000, 10000) filter isPrime) apply 1

 -->  (if (1000 >= 10000) empty              // by expanding streamRange
       else cons(1000, streamRange(1000 + 1, 10000))
      .filter(isPrime).apply(1)

 -->  cons(1000, streamRange(1000 + 1, 10000))  // by evaluating if
      .filter(isPrime).apply(1)
```

Evaluation Trace (2)

Let's abbreviate `cons(1000, streamRange(1000 + 1, 10000))` to `C1`.

```
C1.filter(isPrime).apply(1)
```

Evaluation Trace (2)

Let's abbreviate `cons(1000, streamRange(1000 + 1, 10000))` to `C1`.

```
      C1.filter(isPrime).apply(1)

-->   (if (C1.isEmpty) C1                          // by expanding filter
       else if (isPrime(C1.head)) cons(C1.head, C1.tail.filter(isPrime))
       else C1.tail.filter(isPrime))
      .apply(1)
```

Evaluation Trace (2)

Let's abbreviate `cons(1000, streamRange(1000 + 1, 10000))` to `C1`.

```
     C1.filter(isPrime).apply(1)

-->  (if (C1.isEmpty) C1                            // by expanding filter
      else if (isPrime(C1.head)) cons(C1.head, C1.tail.filter(isPrime))
      else C1.tail.filter(isPrime))
     .apply(1)

-->  (if (isPrime(C1.head)) cons(C1.head, C1.tail.filter(isPrime))
      else C1.tail.filter(isPrime))                 // by eval. if
     .apply(1)
```

Evaluation Trace (2)

Let's abbreviate `cons(1000, streamRange(1000 + 1, 10000))` to `C1`.

```
      C1.filter(isPrime).apply(1)

-->   (if (C1.isEmpty) C1                    // by expanding filter
       else if (isPrime(C1.head)) cons(C1.head, C1.tail.filter(isPrime))
       else C1.tail.filter(isPrime))
      .apply(1)

-->   (if (isPrime(C1.head)) cons(C1.head, C1.tail.filter(isPrime))
       else C1.tail.filter(isPrime))         // by eval. if
      .apply(1)

-->   (if (isPrime(1000)) cons(C1.head, C1.tail.filter(isPrime))
       else C1.tail.filter(isPrime))         // by eval. head
      .apply(1)
```

Evaluation Trace (3)

```
-->> (if (false) cons(C1.head, C1.tail.filter(isPrime))  // by eval. isPrime
      else C1.tail.filter(isPrime))
     .apply(1)
```

Evaluation Trace (3)

```
-->> (if (false) cons(C1.head, C1.tail.filter(isPrime))  // by eval. isPrime
      else C1.tail.filter(isPrime))
     .apply(1)

-->  C1.tail.filter(isPrime).apply(1)                    // by eval. if
```

Evaluation Trace (3)

```
-->> (if (false) cons(C1.head, C1.tail.filter(isPrime))  // by eval. isPrime
      else C1.tail.filter(isPrime))
     .apply(1)

-->  C1.tail.filter(isPrime).apply(1)                    // by eval. if

-->> streamRange(1001, 10000)                            // by eval. tail
     .filter(isPrime).apply(1)
```

The evaluation sequence continues like this until:

Evaluation Trace (3)

```
-->> (if (false) cons(C1.head, C1.tail.filter(isPrime))  // by eval. isPrime
      else C1.tail.filter(isPrime))
     .apply(1)

-->  C1.tail.filter(isPrime).apply(1)                    // by eval. if

-->> streamRange(1001, 10000)                            // by eval. tail
     .filter(isPrime).apply(1)
```

The evaluation sequence continues like this until:

```
-->> streamRange(1009, 10000)
     .filter(isPrime).apply(1)

-->  cons(1009, streamRange(1009 + 1, 10000))            // by eval. streamRange
     .filter(isPrime).apply(1)
```

Evaluation Trace (4)

Let's abbreviate `cons(1009, streamRange(1009 + 1, 10000))` to `C2`.

```
C2.filter(isPrime).apply(1)
```

Evaluation Trace (4)

Let's abbreviate `cons(1009, streamRange(1009 + 1, 10000))` to `C2`.

```
    C2.filter(isPrime).apply(1)

--> cons(1009, C2.tail.filter(isPrime)).apply(1)      // filter
```

Evaluation Trace (4)

Let's abbreviate `cons(1009, streamRange(1009 + 1, 10000))` to `C2`.

```
     C2.filter(isPrime).apply(1)

 --> cons(1009, C2.tail.filter(isPrime)).apply(1)

 --> if (1 == 0) cons(1009, C2.tail.filter(isPrime)).head // by eval. apply
     else cons(1009, C2.tail.filter(isPrime)).tail.apply(0)
```

Assuming `apply` is defined like this in `Stream[T]`:

```
def apply(n: Int): T =
  if (n == 0) head
  else tail.apply(n-1)
```

Evaluation Trace (4)

Let's abbreviate `cons(1009, streamRange(1009 + 1, 10000))` to `C2`.

```
      C2.filter(isPrime).apply(1)

-->> cons(1009, C2.tail.filter(isPrime)).apply(1)         // by eval. filter

-->  if (1 == 0) cons(1009, C2.tail.filter(isPrime)).head // by eval. apply
     else cons(1009, C2.tail.filter(isPrime)).tail.apply(0)

-->  cons(1009, C2.tail.filter(isPrime)).tail.apply(0)    // by eval. if
```

Evaluation Trace (4)

Let's abbreviate `cons(1009, streamRange(1009 + 1, 10000))` to `C2`.

```
      C2.filter(isPrime).apply(1)

-->> cons(1009, C2.tail.filter(isPrime)).apply(1)        // by eval. filter

-->  if (1 == 0) cons(1009, C2.tail.filter(isPrime)).head // by eval. apply
     else cons(1009, C2.tail.filter(isPrime)).tail.apply(0)

-->  cons(1009, C2.tail.filter(isPrime)).tail.apply(0)    // by eval. if

-->  C2.tail.filter(isPrime).apply(0)                     // by eval. tail
```

Evaluation Trace (4)

Let's abbreviate `cons(1009, streamRange(1009 + 1, 10000))` to `C2`.

```
      C2.filter(isPrime).apply(1)

-->> cons(1009, C2.tail.filter(isPrime)).apply(1)          // by eval. filter

-->  if (1 == 0) cons(1009, C2.tail.filter(isPrime)).head // by eval. apply
     else cons(1009, C2.tail.filter(isPrime)).tail.apply(0)

-->  cons(1009, C2.tail.filter(isPrime)).tail.apply(0)     // by eval. if

-->  C2.tail.filter(isPrime).apply(0)                       // by eval. tail

-->  streamRange(1010, 10000).filter(isPrime).apply(0)     // by eval. tail
```

Evaluation Trace (5)

The process continues until

```
      ...
-->   streamRange(1013, 10000).filter(isPrime).apply(0)
```

Evaluation Trace (5)

The process continues until

```
      ...
-->   streamRange(1013, 10000).filter(isPrime).apply(0)

-->   cons(1013, streamRange(1013 + 1, 10000))          // by eval. streamRange
      .filter(isPrime).apply(0)
```

Let `C3` be a shorthand for `cons(1013, streamRange(1013 + 1, 10000)`.

```
==    C3.filter(isPrime).apply(0)
```

Evaluation Trace (5)

The process continues until

```
      ...
 -->  streamRange(1013, 10000).filter(isPrime).apply(0)

 -->  cons(1013, streamRange(1013 + 1, 10000))          // by eval. streamRange
      .filter(isPrime).apply(0)
```

Let `C3` be a shorthand for `cons(1013, streamRange(1013 + 1, 10000)`.

```
 ==   C3.filter(isPrime).apply(0)

 -->> cons(1013, C3.tail.filter(isPrime)).apply(0)      // by eval. filter
```

Evaluation Trace (5)

The process continues until

```
      ...
 -->  streamRange(1013, 10000).filter(isPrime).apply(0)

 -->  cons(1013, streamRange(1013 + 1, 10000))          // by eval. streamRange
      .filter(isPrime).apply(0)
```

Let `C3` be a shorthand for `cons(1013, streamRange(1013 + 1, 10000)`.

```
 ==   C3.filter(isPrime).apply(0)

 -->> cons(1013, C3.tail.filter(isPrime)).apply(0)      // by eval. filter

 -->  1013                                              // by eval. apply
```

Only the part of the stream necessary to compute the result has been constructed.

Computing with Infinite Sequences

Infinite Streams

You saw that all elements of a stream except the first one are computed only when they are needed to produce a result.

This opens up the possibility to define infinite streams!

For instance, here is the stream of all integers starting from a given number:

```
def from(n: Int): Stream[Int] = n #:: from(n+1)
```

The stream of all natural numbers:

Infinite Streams

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```

The stream of all natural numbers:

```
val nats = from(0)
```

The stream of all multiples of 4:

Infinite Streams

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```

The stream of all natural numbers:

```
val nats = from(0)
```

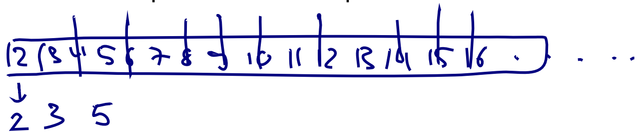
The stream of all multiples of 4:

```
nats map (_ * 4)
```

The Sieve of Eratosthenes

The Sieve of Eratosthenes is an ancient technique to calculate prime numbers.

The idea is as follows:



- Start with all integers from 2, the first prime number.
- Eliminate all multiples of 2.
- The first element of the resulting list is 3, a prime number.
- Eliminate all multiples of 3.
- Iterate forever. At each step, the first number in the list is a prime number and we eliminate all its multiples.

The Sieve of Eratosthenes in Code

Here's a function that implements this principle:

```
def sieve(s: Stream[Int]): Stream[Int] =
  s.head #:: sieve(s.tail filter (_ % s.head != 0))

val primes = sieve(from(2))
```

To see the list of the first `N` prime numbers, you can write

```
(primes take N).toList
```

Back to Square Roots

Our previous algorithm for square roots always used a `isGoodEnough` test to tell when to terminate the iteration.

With streams we can now express the concept of a converging sequence without having to worry about when to terminate it:

```
def sqrtStream(x: Double): Stream[Double] = {
  def improve(guess: Double) = (guess + x / guess) / 2
  lazy val guesses: Stream[Double] = 1 #:: (guesses map improve)
  guesses
}
```

Termination

We can add `isGoodEnough` later.

```
def isGoodEnough(guess: Double, x: Double) =
  math.abs((guess * guess - x) / x) < 0.0001

sqrtStream(4) filter (isGoodEnough(_, 4))
```

Exercise:

Consider two ways to express the infinite stream of multiples of a given number N:

```
val xs = from(1) map (_ * N)

val ys = from(1) filter (_ % N == 0)
```

Which of the two streams generates its results faster?

```
O         from(1) map (_ * N)
O         from(1) filter (_ % N == 0)
```

Exercise:

Consider two ways to express the infinite stream of multiples of a given number N:

```
val xs = from(1) map (_ * N)

val ys = from(1) filter (_ % N == 0)
```

N = 3

1 2 3 4 5
3 6 9 12 ....

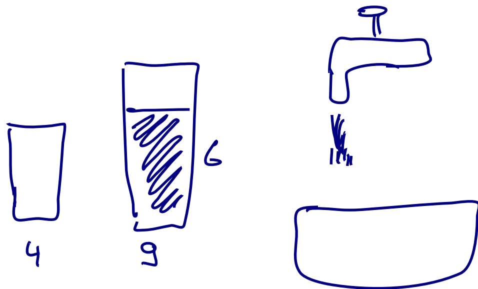
1 2 3 4 5 6 7
3 6 9 12

Which of the two streams generates its results faster?

- ● `from(1) map (_ * N)`
- O `from(1) filter (_ % N == 0)`

Case Study

The Water Pouring Problem



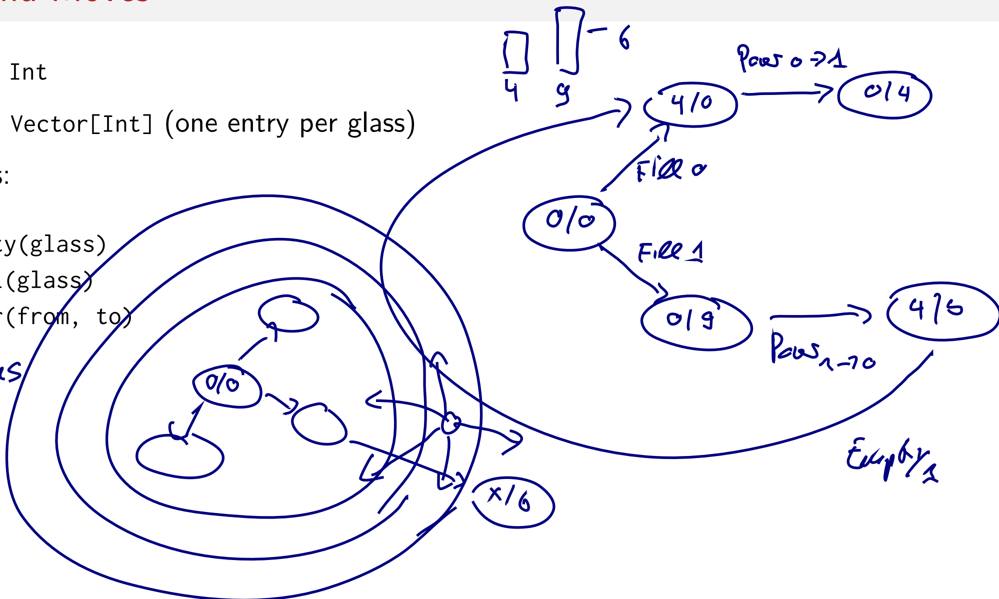
States and Moves

Glass: `Int`

State: `Vector[Int]` (one entry per glass)

Moves:

```
Empty(glass)
Fill(glass)
Pour(from, to)
```



Variants

In a program of the complexity of the pouring program, there are many choices to be made.

Choice of representations.

- Specific classes for moves and paths, or some encoding?
- Object-oriented methods, or naked data structures with functions?

The present elaboration is just one solution, and not necessarily the shortest one.

Guiding Principles for Good Design

- Name everything you can.
- Put operations into natural scopes.
- Keep degrees of freedom for future refinements.

Course Conclusion

Traits of Functional Programming

Functional programming provides a coherent set of notations and methods based on

- higher-order functions,
- case classes and pattern matching,
- immutable collections,
- absence of mutable state,
- flexible evaluation strategies: *strict/lazy/by name*.

A useful toolkit for every programmer.

A different way of thinking about programs.

More Material on Scala

Reference material:

Scala Ref Card (adapted from a forum post by Laurent Poulain)

Twitter's Scala School

Programming in Scala

Scala Tour

To stay current:

Scala Meetups

Typesafe Blog and Newsletter

This Week in Scala Blogs

What Remains to Be Covered

Worthwhile topics we did not cover in this course:

Functional programming and state

- what does it mean to have mutable state?
- what changes if we add it?

Parallelism

- how to exploit immutability for parallel execution.

Domain-Specific Languages

- high-level libraries as embedded DSLs.
- interpretation techniques for external DSLs.