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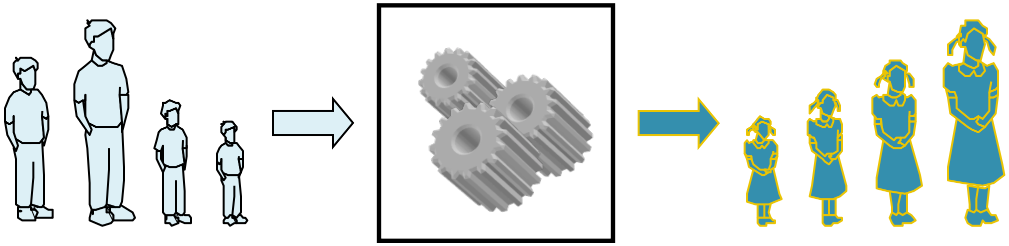
Data Structures
Lecture 8

Recap

- What should you have learned?
  - Basic java programming skills
    - Object-oriented programming
    - Classes and objects
    - Inheritance, exception handling, generics
    - Java class library
  - Basic data structures and their applications
    - Linear data structure: linked list, array, stack, queue
    - Hierarchical data structures: tree and heap

Wrap up

- What are you going to learn in the rest of this semester ?
  - Algorithms
    - Analysis of algorithms
    - Brute force, divide and conquer, dynamic programming
    - Sorting
  - Advanced data structures
    - Hash table
    - Map and dictionary
    - Graph



Input **Algorithm** **Output**

Analysis of Algorithms

How good is your program?

Running Time

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics

Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a method like `System.currentTimeMillis()` to get an accurate measure of the actual running time
- Plot the results

Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used

Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, n.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

Pseudo code

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Example: find the max element of an array

```
Algorithm arrayMax(A, n)
    Input array A of n integers
    Output the maximum element of A

    currentMax ← A[0]
    for i ← 1 to n - 1 do
        if A[i] > currentMax then
            currentMax ← A[i]
    return currentMax
```

Pseudo code

Example: find the max element of an array

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Algorithm arrayMax(A, n)
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        if A[i] > currentMax then
            currentMax ← A[i]
    return currentMax
```

Find the min element of an array

```
Algorithm arrayMin(A, n)
    Input array A of n integers
    Output the minimum element of A

    currentMin ← A[0]
    for i ← 1 to n - 1 do
        if A[i] < currentMin then
            currentMin ← A[i]
    return currentMin
```

Pseudo code

Find the min element of an array

```
Algorithm arrayMin($A$, $n$)
    Input array $A$ of $n$ integers
    Output the minimum element of $A$

    $currentMin \leftarrow A[0]$
    for $i \leftarrow 1$ to $n - 1$ do
        if $A[i] < currentMin$ then
            $currentMin \leftarrow A[i]$
    return $currentMin$
```

Sum all the elements of an array

```
Algorithm arraySum($A$, $n$)
    Input array $A$ of $n$ integers
    Output sum of all the elements of $A$

    $currentSum \leftarrow 0$
    for $i \leftarrow 0$ to $n - 1$ do
        $currentSum \leftarrow currentSum+A[i]$
    return $currentSum$
```

Pseudo code

Sum all the elements of an array

```
Algorithm arraySum(A, n)
    Input array A of n integers
    Output sum of all the elements of A

    currentSum ← 0
    for i ← 0 to n - 1 do
        currentSum ← currentSum+A[i]
    return currentSum
```

Multiply all the elements of an array

```
Algorithm arrayMultiply(A, n)
    Input array A of n integers
    Output Multiply all the elements of A

    current ← 1
    for i ← 0 to n - 1 do
        current ← current*A[i]
    return current
```

Pseudo code Details

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

```
Algorithm method (arg [, arg…])
    Input …
    Output …
```

Pseudo code Details

- Method call

  var.method (***arg*** [, ***arg***…])

- Return value

  return ***expression***

- Expressions

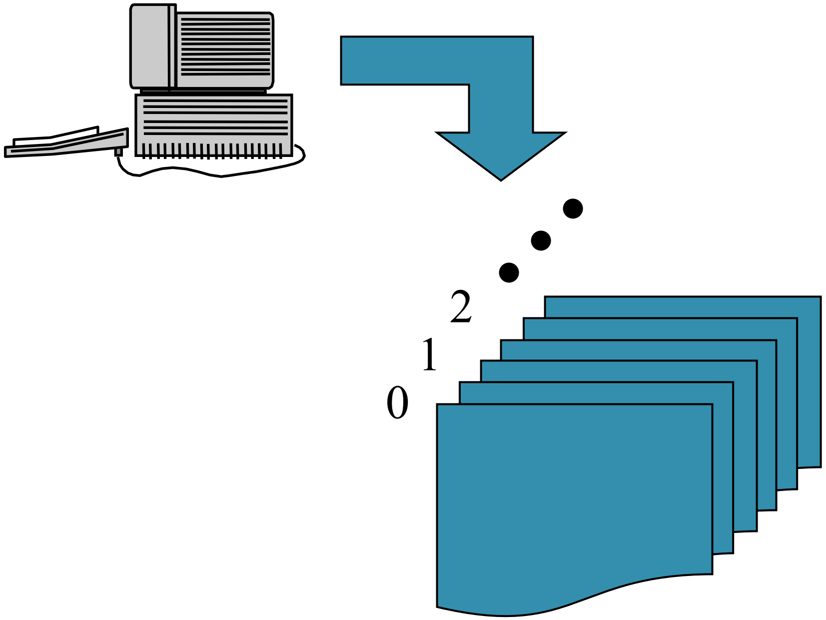
  ← Assignment (like = in Java)

  = Equality testing (like == in Java)

  n^2 Superscripts and other mathematical formatting allowed

The Random Access Machine (RAM) Model

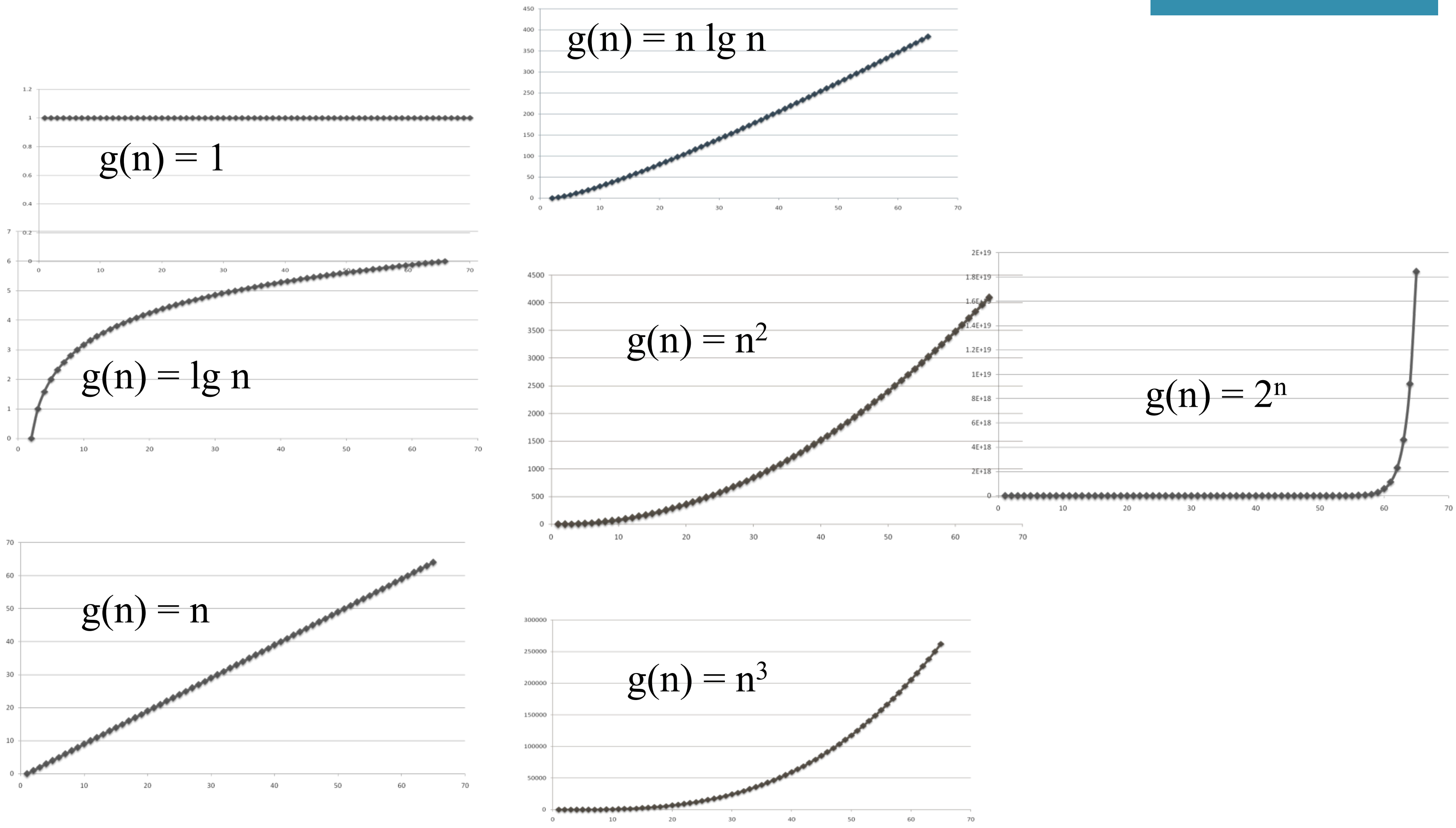
- A **CPU**
- An potentially unbounded bank of **memory** cells, each of which can hold an arbitrary number or character
- Memory cells are numbered and accessing any cell in memory takes unit time.



Seven Important Functions

- Seven functions that often appear in algorithm analysis:
  - Constant ≈ $\mathbf{1}$
  - Logarithmic ≈ $\log \boldsymbol{n}$
  - Linear ≈ $\boldsymbol{n}$
  - N-Log-N ≈ $\boldsymbol{n} \log \boldsymbol{n}$
  - Quadratic ≈ $\boldsymbol{n}^2$
  - Cubic ≈ $\boldsymbol{n}^3$
  - Exponential ≈ $\mathbf{2}^n$

Functions Graphed Using “Normal” Scale



Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important (we will see why later)
- Assumed to take a constant amount of time in the RAM model
- Examples:
    - Evaluating an expression
    - Assigning a value to a variable
    - Indexing into an array
    - Calling a method
    - Returning from a method

Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

```
Algorithm arrayMax(A, n)                          # operations
    currentMax ← A[0]                                   2
    for i ← 1 to n − 1 do                               2n
        if A[i] > currentMax then                    2(n − 1)
            currentMax ← A[i]                        2(n − 1)
    { increment counter i }                          2(n − 1)
    return currentMax                                   1

                                          Total      8n − 2
```

Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

```
Algorithm arrayMultiply(A, n)                 #operations
  current ← 1                                 1
  for i ← 0 to n - 1 do                       2(n+1)
    current ← current*A[i]                   3n
{ increment counter i }                       2n
  return current                              1

                                   Total   7n+4 =>O(n)
```

Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

```
Algorithm arrayAverage(A, n)                  #operations
  current ← 0                                   1
  for i ← 0 to n - 1 do                         2(n+1)
    current ← current+A[i]                      3n
{ increment counter i }                         2n
  return current/n                              2

                                    Total   7n+5 =>O(n)
```

Estimating Running Time

- Algorithm ***arrayMax*** executes $8n - 2$ primitive operations in the worst case. Define:

  a = Time taken by the fastest primitive operation

  b = Time taken by the slowest primitive operation

- Let $T(n)$ be worst-case time of ***arrayMax.*** Then

  $$a\,(8n - 2) \leq T(n) \leq b(8n - 2)$$

- Hence, the running time $T(n)$ is bounded by two linear functions

Growth Rate of Running Time

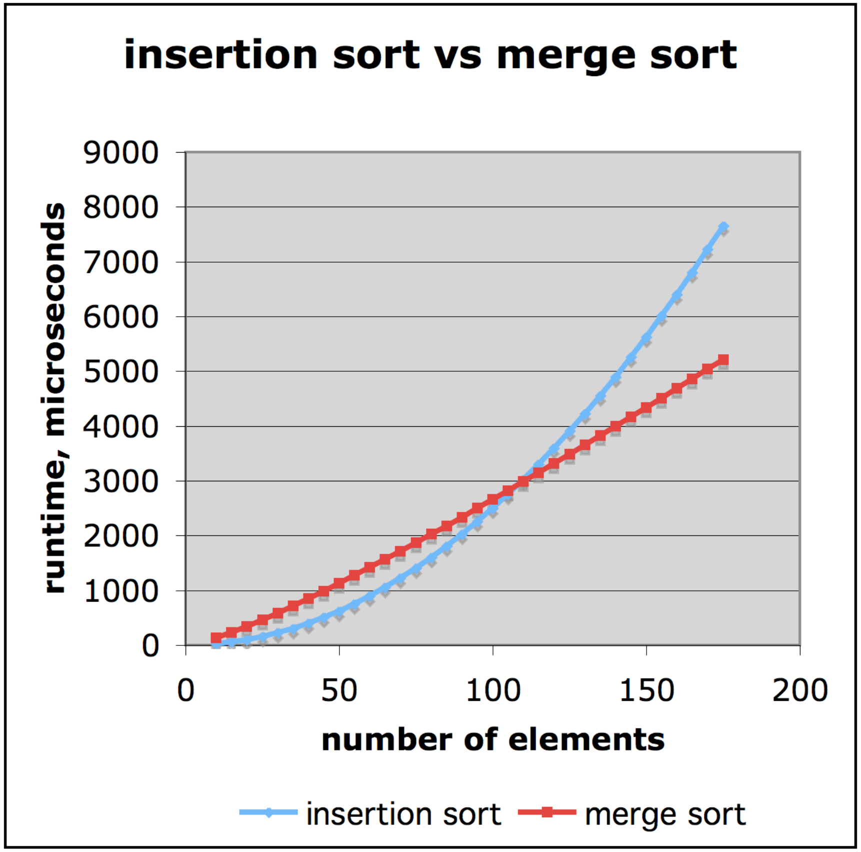
- Changing the hardware/ software environment
    - Affects $T(n)$ by a constant factor, but
    - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm ***arrayMax***

Why Growth Rate Matters

if runtime is...	time for n + 1	time for 2 n	time for 4 n
$c \lg n$	$c \lg (n + 1)$	$c (\lg n + 1)$	$c(\lg n + 2)$
$c\, n$	$c\, (n + 1)$	$2c\, n$	$4c\, n$
$c\, n \lg n$	$\sim c\, n \lg n + c\, n$	$2c\, n \lg n + 2cn$	$4c\, n \lg n + 4cn$
$c\, n^2$	$\sim c\, n^2 + 2c\, n$	**$4c\, n^2$**	$16c\, n^2$
$c\, n^3$	$\sim c\, n^3 + 3c\, n^2$	$8c\, n^3$	$64c\, n^3$
$c\, 2^n$	$c\, 2^{n+1}$	$c\, 2^{2n}$	$c\, 2^{4n}$

runtime quadruples when problem size doubles

Comparison of Two Algorithms



insertion sort is
$n^2 / 4$

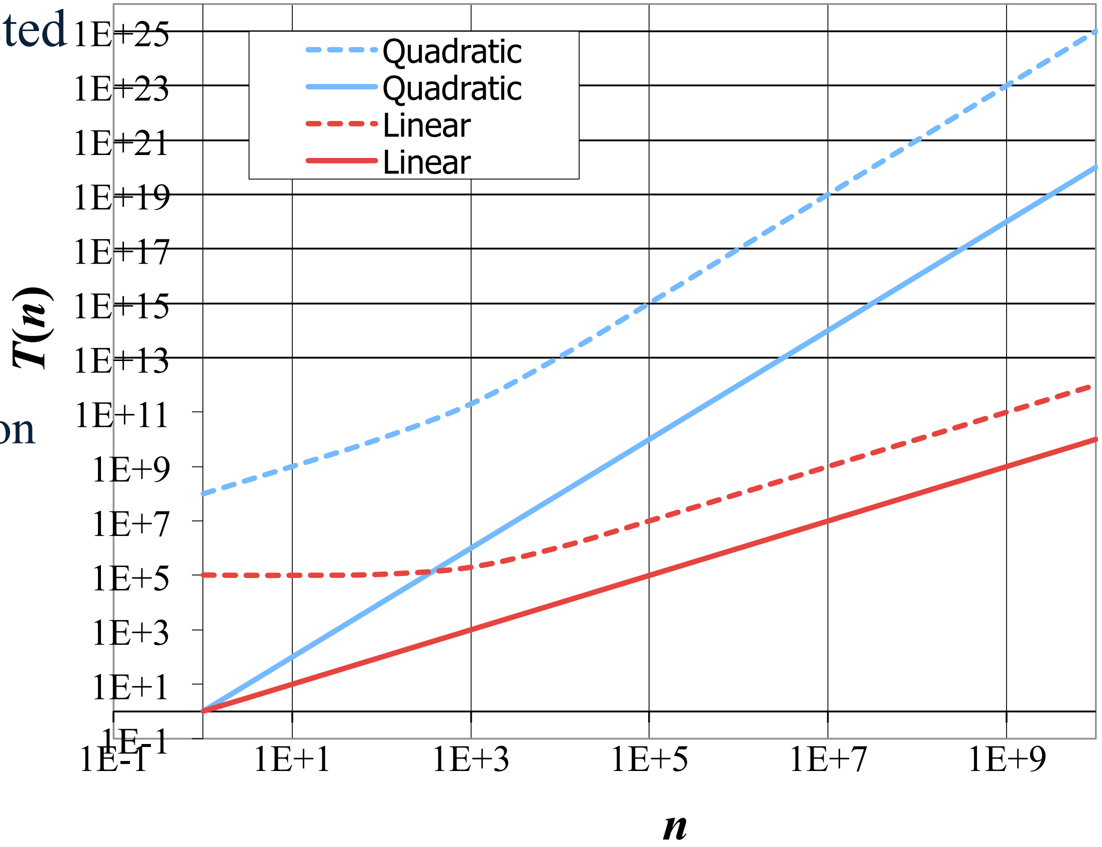
merge sort is
$2 n \lg n$

sort a million items?
insertion sort takes
roughly 70 hours
while
merge sort takes
roughly 40 seconds

This is a slow machine, but if 100 x as fast then it's 40 minutes versus less than 0.5 seconds

Constant Factors

- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 \boldsymbol{n} + 10^5$ is a linear function
  - $10^5 \boldsymbol{n}^2 + 10^8 \boldsymbol{n}$ is a quadratic function

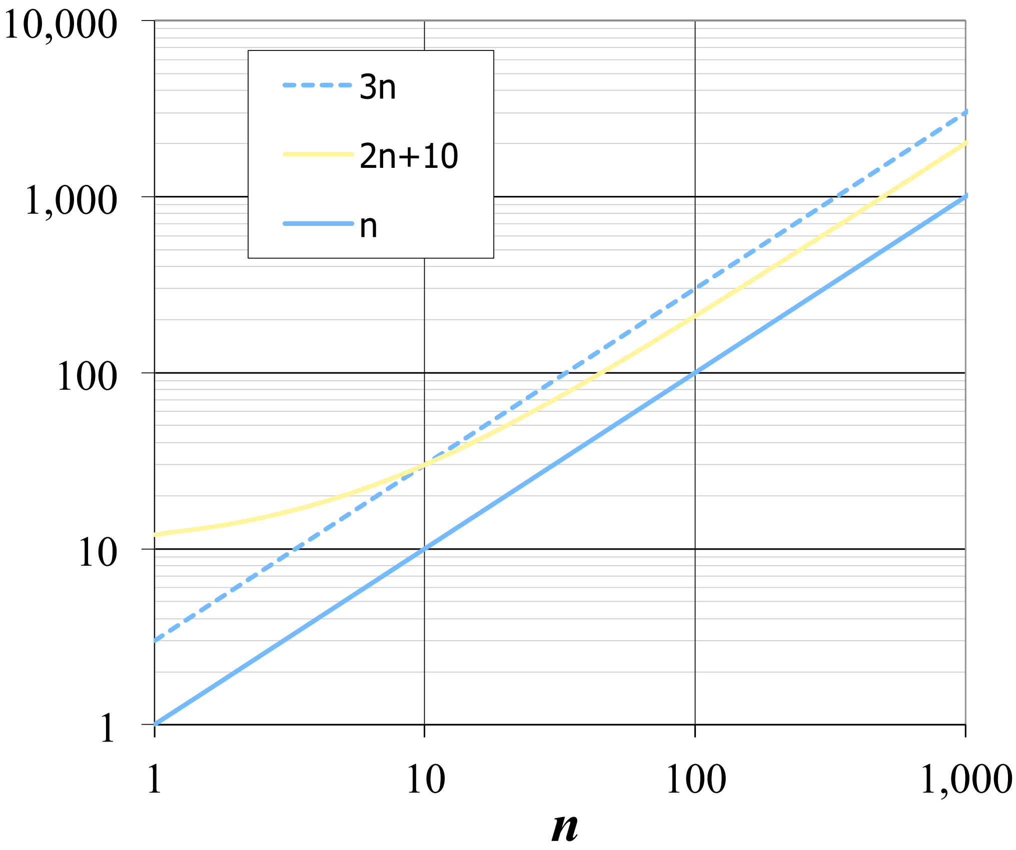


Big-Oh Notation

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

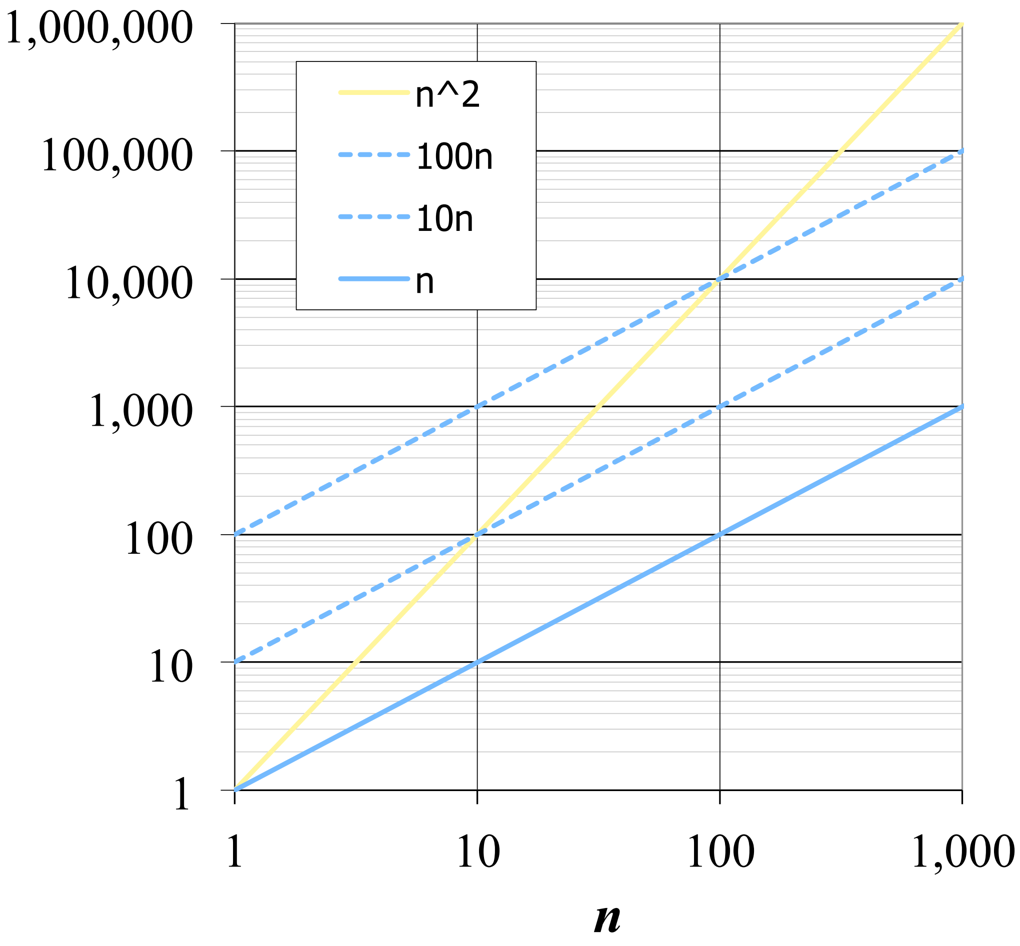
$$f(n) \leq cg(n) \quad \text{for } n \geq n_0$$

- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \leq cn$
  - $(c - 2)\, n \geq 10$
  - $n \geq 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$



Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since c must be a constant



More Big-Oh Example

- $7n-2$ is $O(n)$
  - need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \cdot n$ for $n \geq n_0$
  - this is true for $c = 7$ and $n_0 = 1$
- $3n^3 + 20n^2 + 5$ is $O(n^3)$
  - need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \cdot n^3$ for $n \geq n_0$
  - this is true for $c = 4$ and $n_0 = 21$
- $3 \log n + 5$ is $O(\log n)$
  - need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \cdot \log n$ for $n \geq n_0$
  - this is true for $c = 8$ and $n_0 = 2$

Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

Big-Oh Rules

- If $f(n)$ is a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
    1. Drop lower-order terms
    2. Drop constant factors
- Use the smallest possible class of functions
    - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
    - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Algorithm Analysis

- The asymptotic analysis of an algorithm determines the running time in big-Oh notation

- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation

Asymptotic Algorithm Analysis

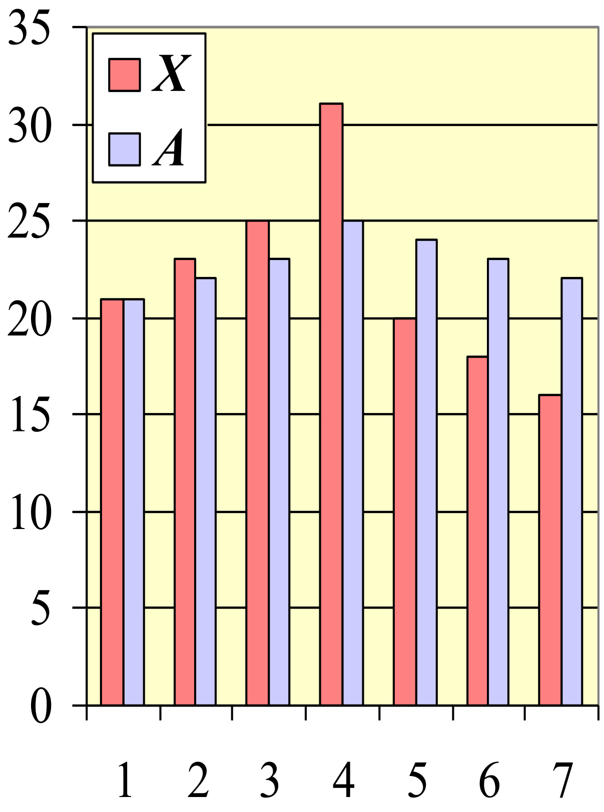
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $8n - 2$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The ***i***-th prefix average of an array ***X*** is average of the first (***i*** + 1) elements of ***X***:

$$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$$

- Computing the array ***A*** of prefix averages of another array ***X*** has applications to financial analysis



Exercise

- Implement prefixAverage
- **Input:**
  - Get n integers from a txt file
  - The first integer indicates the number of integers (the size of X)
- **Output:**
  - Print out a sequence of integers
  - The ith integer indicates the average of the first i+1 input numbers (starting from the second input)

Input: 4 1 2 3 5

Output: 1 2 2

Prefix Average (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

```
Algorithm prefixAverages1(X, n)
    Input array X of n integers
    Output array A of prefix averages of X            #operations
    A ← new array of n integers                              n
    for i ← 0 to n − 1 do                                    n
        s ← X[0]                                             n
        for j ← 1 to i do                           1 + 2 + …+ (n − 1)
            s ← s + X[j]                            1 + 2 + …+ (n − 1)
        A[i] ← s / (i + 1)                                   n
    return A                                                 1
```

Arithmetic Progression

- The running time of ***prefixAverages1*** is $\boldsymbol{O}(1 + 2 + \ldots + \boldsymbol{n})$
- The sum of the first $\boldsymbol{n}$ integers is $\boldsymbol{n}(\boldsymbol{n} + 1) / 2$
    - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverages1*** runs in $\boldsymbol{O}(\boldsymbol{n}^2)$ time

Prefix Average (Linear)

- The following algorithm computes prefix averages in linear time by keeping a running sum
- Algorithm ***prefixAverages2*** runs in ***O***(***n***) time

```
Algorithm prefixAverages2($X$, $n$)
    Input array $X$ of $n$ integers
    Output array $A$ of prefix averages of $X$      #operations
    $A \leftarrow$ new array of $n$ integers              $n$
    $s \leftarrow 0$                                      $1$
    for $i \leftarrow 0$ to $n - 1$ do                        $n$
      $s \leftarrow s + X[i]$                             $n$
      $A[i] \leftarrow s / (i + 1)$                       $n$
    return $A$                                        $1$
```

Relatives of Big-Oh

- big-Omega
  - f(n) is $\Omega(g(n))$ if there is a constant $c > 0$
  - and an integer constant $n_0 \geq 1$ such that
  - $f(n) \geq c \cdot g(n)$ for $n \geq n_0$

- big-Theta
  - f(n) is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c' \cdot g(n) \leq f(n) \leq c'' \cdot g(n)$ for $n \geq n_0$

Intuition for Asymptotic Notation

- Big-Oh
  - f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)
- big-Omega
  - f(n) is Ω(g(n)) if f(n) is asymptotically **greater than or equal** to g(n)
- big-Theta
  - f(n) is Θ(g(n)) if f(n) is asymptotically **equal** to g(n)

Examples of Using Relatives of Big-Oh

- **$5n^2$ is $\Omega(n^2)$**
  - $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$
  - let $c = 5$ and $n_0 = 1$

- **$5n^2$ is $\Omega(n)$**
  - $f(n)$ is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $\mathrm{f}(n) \geq c \bullet g(n)$ for $n \geq n_0$
  - let $c = 1$ and $n_0 = 1$

- **$5n^2$ is $\Theta(n^2)$**
  - $f(n)$ is $\Theta(g(n))$ if it is $\Omega(n^2)$ and $O(n^2)$. We have already seen the former, for the latter recall that $f(n)$ is $O(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $\mathrm{f}(n) \leq c \bullet \mathrm{g}(n)$ for $n \geq n_0$
  - Let $c = 5$ and $n_0 = 1$

Coming Up…

- For more about Big O: Read Text Book 4
- Next week we will talk about “Divide and Conquer/ Sorting”: Read Text Book 11