Fall 2019

Data Structures Lecture 10

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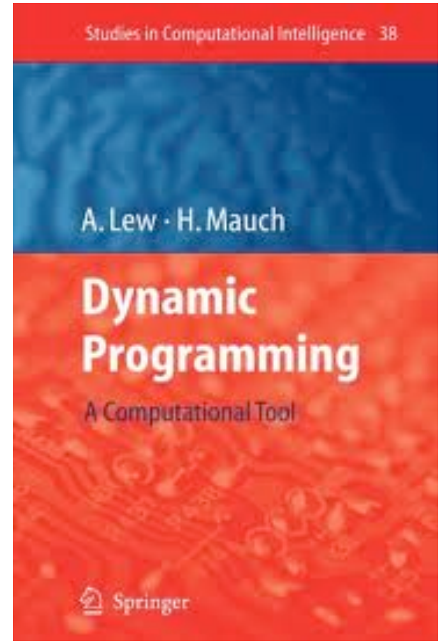
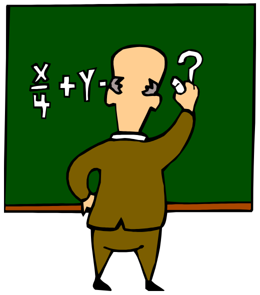


Fundamental Algorithms

Brute force, Greedy, Dynamic Programming:

Dynamic Programming Technique

- Primarily for optimization problems
- Applies to a problem that at first seems to require a lot of time (possibly exponential), provided we have:
  - **Simple subproblems:** the subproblems can be defined in terms of a few variables, such as j, k, l, m, and so on.
  - **Subproblem optimality:** the global optimum value can be defined in terms of optimal subproblems
  - **Subproblem overlap:** the subproblems are not independent, but instead they overlap (hence, should be constructed bottom-up).

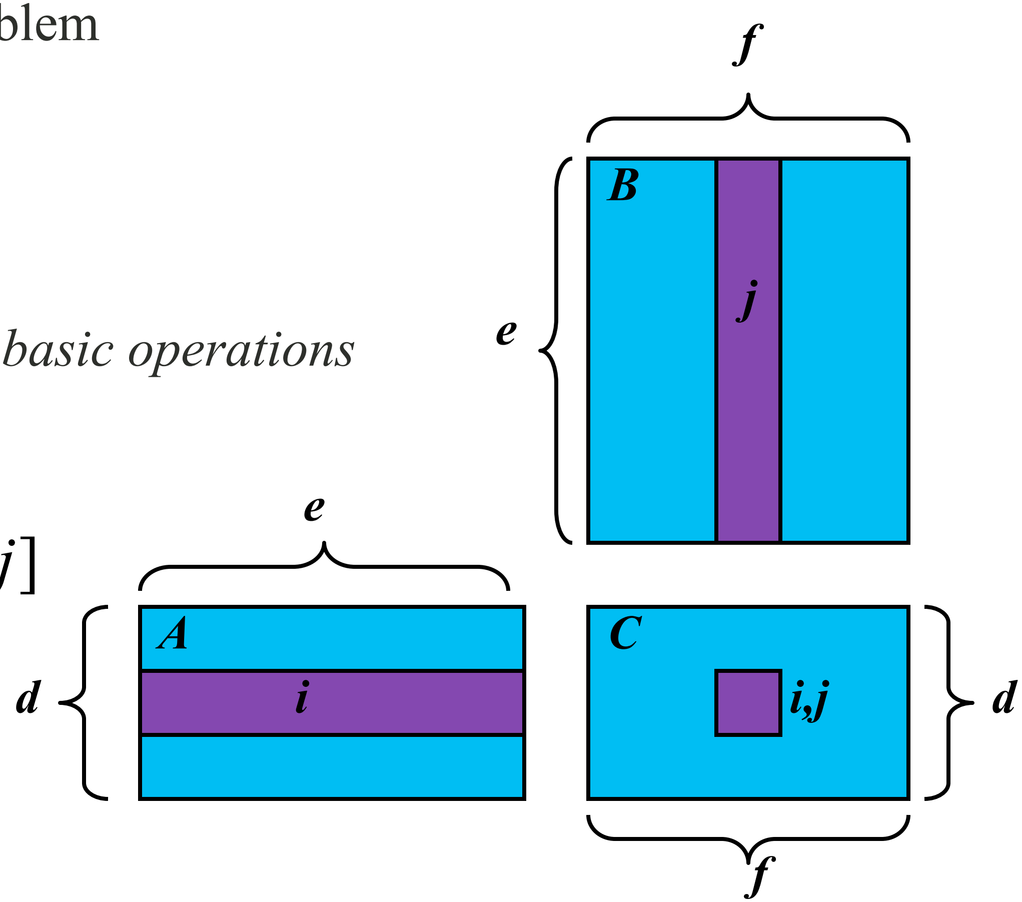


Matrix Chain-Products

Lets start from a mathematic problem

- Matrix Multiplication.
  - $C = A*B$
  - A is $d \times e$ and B is $e \times f$
  - *$A*B$ takes $d \times e \times f$ times of basic operations*

$$C[i,j] = \sum_{k=0}^{e-1} A[i,k] * B[k,j]$$



Matrix Chain-Products

- Compute $A=A_0*A_1*\ldots*A_{n-1}$
- A_i is $d_i \times d_{i+1}$
- Problem: We want to find a way to compute the result with the minimal number of operations

Matrix Chain-Products

- How to put parentheses on matrix?
- Example:
  - B is 3×100
  - C is 100×5
  - D is 5×5
  - (B*C)*D takes 1500 + 75 = 1575 ops
  - B*(C*D) takes 1500 + 2500 = 4000 ops
  - The order of computation matters!
  - We want to figure out the way with the minimal cost

Brute-force

- An enumeration approach
- **Matrix Chain-Product Alg.:**
  - Try all possible ways to parenthesize $A=A_0*A_1*\ldots*A_{n-1}$
  - Calculate number of ops for each one
  - Pick the one that is best
- Running time:
  - The number of paranethesizations is equal to the number of binary trees with n nodes
  - This is **exponential**!
  - It is called the Catalan number, and it is almost 4^n.
  - This is a terrible algorithm!

Greedy

- Choose the local optimal iteratively
- Repeatedly select the product that uses the fewest operations.
- Example:
  - A is 10×5
  - B is 5×10
  - C is 10×5
  - D is 5×10
  - A*B or B*C or C*D → B*C
  - A*((B*C)*D) takes 500+250+250 = 1000 ops

Another example

- Another example
  - A is 101×11
  - B is 11×9
  - C is 9×100
  - D is 100×99
- The greedy approach gives A*((B*C)*D)), which takes 109989+9900+108900=228789 ops
- However, (A*B)*(C*D) takes 9999+89991+89100=189090 ops
- This is a counter example that the greedy approach does not give us an optimal solution

Dynamic Programming

- Simplifying a complicated problem by breaking it down into simpler sub-problems in a recursive manner

Two key observations:

- The problem can be split into sub-problems
- The optimal solution can be defined in terms of optimal sub-problems

Dynamic Programming

- Find the best parenthesization of A_i*A_{i+1}*…*A_j.
- Let $N_{i,j}$ denote the number of operations done by this subproblem.
- The optimal solution for the whole problem is $N_{0,n-1}$.
- There has to be a final multiplication (root of the expression tree) for the optimal solution.
- Say, the final multiply is at index i: (A_0*…*A_i)*(A_{i+1}*…*A_{n-1}).

Dynamic Programming

- Then the optimal solution $N_{0,n-1}$ is the sum of two optimal subproblems, $N_{0,i}$ and $N_{i+1,n-1}$ plus the time for the last multiply
- If the global optimum did not have these optimal subproblems, we could define an even better “optimal” solution.
- We can compute $N_{i,j}$ by considering each k

A Characterizing Equation

- Let us consider all possible places for that final multiply:
  - Recall that A_i is a $d_i \times d_{i+1}$ dimensional matrix.
  - So, a characterizing equation for $N_{i,j}$ is the following:

$$N_{i,j} = \min_{i \le k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$

- Note that sub-problems **overlap** and hence we cannot divide the problem into completely independent sub-problems (divide and conquer)

Bottom-up computation

- $N(i,i) = 0$
- $N(i,i+1) = N(i,i) + N(i+1,i+1) + d_i d_{i+1} d_{i+2}$
- $N(i,i+2) = \min \{$

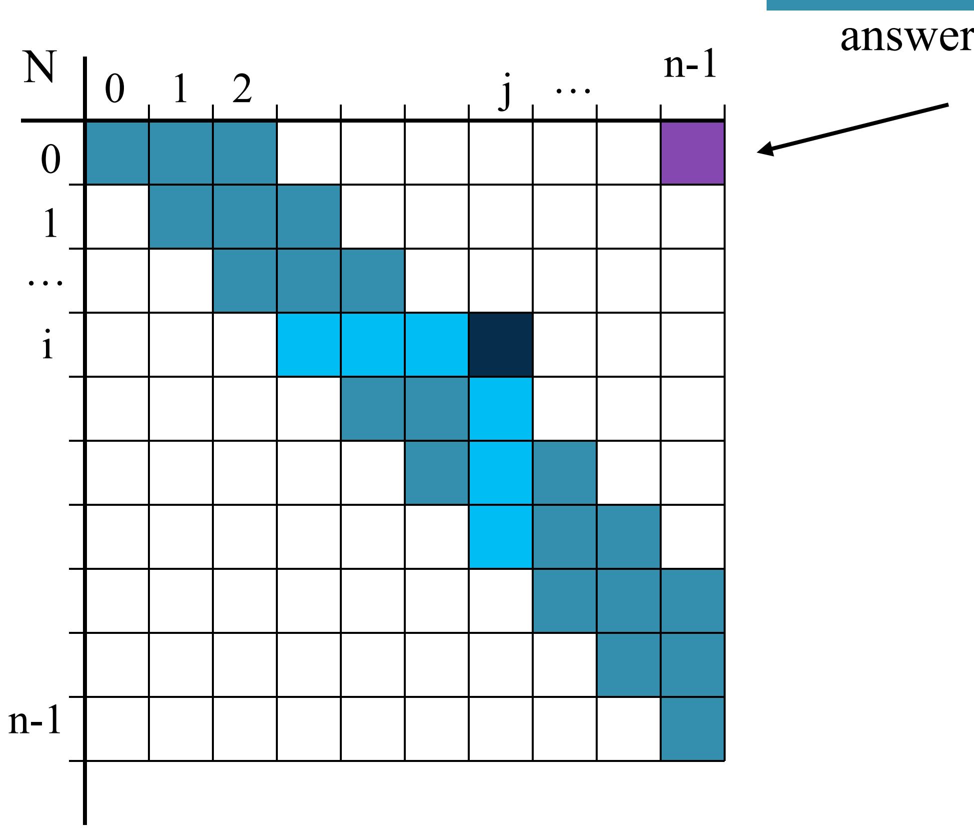
  $N(i,i) + N(i+1,i+2) + d_i d_{i+1} d_{i+2}$

  $N(i,i+1) + N(i+2,i+2) + d_i d_{i+2} d_{i+2}$

  $\}$
- $N(i,i+3) \ldots$
- Until you get $N(i,j)$

A Dynamic Programming Algorithm Visualization

- The bottom-up construction fills in the N array by diagonals
- $N_{i,j}$ gets values from pervious entries in i-th row and j-th column
- Filling in each entry in the N table takes O(n) time.
- Total run time: $O(n^3)$
- Getting actual parenthesization can be done by remembering “k” for each N entry



$$N_{i,j} = \min_{i \le k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$

A Dynamic Programming Algorithm

- Since subproblems overlap, we don't use recursion.
- Instead, we construct optimal subproblems "bottom-up."
- $N_{i,i}$'s are easy, so start with them
- Then do length 2,3,… subproblems, and so on.
- The running time is O(n^3)

Algorithm ***matrixChain***(***S***):

Input: sequence ***S*** of ***n*** matrices to be multiplied

Output: number of operations in an optimal paranethization of ***S***

for $i \leftarrow 1$ **to** $n\text{-}1$ **do**

  $N_{i,i} \leftarrow 0$

for $b \leftarrow 1$ **to** $n\text{-}1$ **do**

  for $i \leftarrow 0$ **to** $n\text{-}b\text{-}1$ **do**

    $j \leftarrow i+b$

    $N_{i,j} \leftarrow +$**infinity**

    for $k \leftarrow$ i **to** $j\text{-}1$ **do**

      $N_{i,j} \leftarrow \min\{N_{i,j}, N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1}\}$

Similarity between strings

- A common text processing problem:
  - Two strands of DNA
  - Two versions of source code for the same program
  - diff (a built-in program for comparing text files)

Subsequences

- A ***subsequence*** of a character string $x_0x_1x_2\ldots x_{n-1}$ is a string of the form $x_{i1}x_{i2}\ldots x_{ik}$, where $i_j < i_{j+1}$.
- Not necessary contiguous but taken in order
- Not the same as substring!
- Example String: ABCDEFGHIJK
  - Subsequence: ACEGIJK
  - Subsequence: DFGHK
  - Not subsequence: DAGH

The Longest Common Subsequence (LCS) Problem

- Given two strings X and Y, the longest common subsequence (LCS) problem is to find a longest subsequence common to both X and Y
- Has applications to DNA similarity testing (alphabet is {A,C,G,T})
- Example: ABCDEFG and XZACKDFWGH
  - have ACDFG as a longest common subsequence

A Poor Approach to the LCS Problem

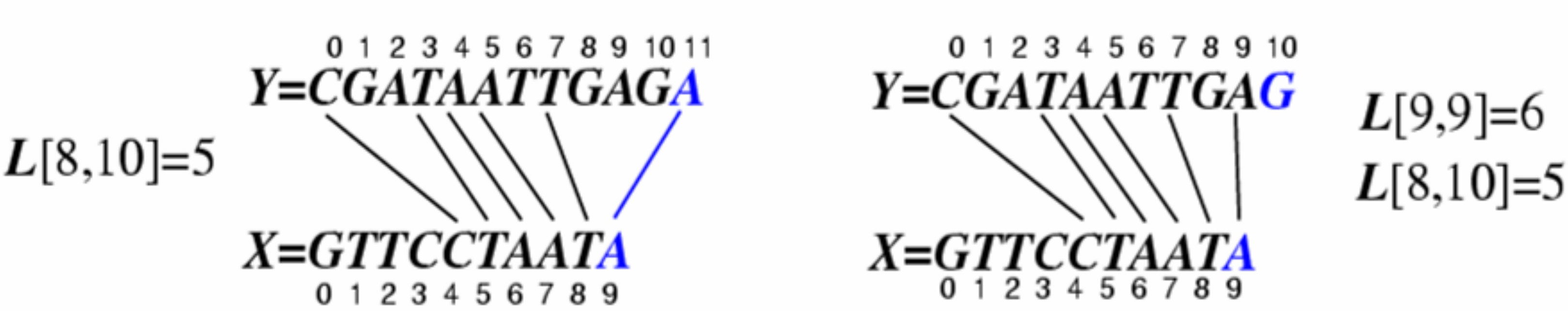
- A Brute-force solution:
  - Enumerate all subsequences of X
  - Test which ones are also subsequences of Y
  - Pick the longest one.
- Analysis:
  - If X is of length n, then it has 2^n subsequences
  - If Y is of length m, the time complexity is $O(2^n m)$
  - This is an exponential-time algorithm!

A Dynamic-Programming Approach to the LCS Problem

- Define L[i,j] to be the length of the longest common subsequence of X[0..i] and Y[0..j].
- Allow for -1 as an index, so L[-1,k] = 0 and L[k,-1]=0, to indicate that the null part of X or Y has no match with the other.
- Then we can define L[i,j] in the general case as follows:
  1. If xi=yj, then L[i,j] = L[i-1,j-1] + 1 (we can add this match)
  2. If xi≠yj, then L[i,j] = max{L[i-1,j], L[i,j-1]} (we have no match here)

A Dynamic-Programming Approach to the LCS Problem

Case 1: Case 2:



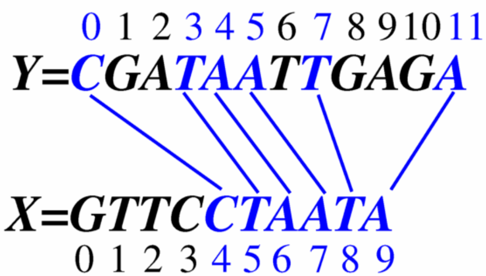
An LCS Algorithm

Algorithm LCS(X,Y):
Input: Strings X and Y with n and m elements, respectively
Output: For i = 0,…,n-1, j = 0,...,m-1, the length L[i, j] of a longest string that is a subsequence of both the string X[0..i] = $x_0x_1x_2\ldots x_i$ and the string Y [0.. j] = $y_0y_1y_2\ldots y_j$

for i =0 to n-1 **do**
    L[i,-1] = 0
for j =0 to m-1 **do**
    L[-1,j] = 0
for i =0 to n-1 **do**
    for j =0 to m-1 **do**
        if $x_i = y_j$ **then**
            L[i, j] = L[i-1, j-1] + 1
        else
            L[i, j] = max{L[i-1, j] , L[i, j-1]}
return array L

Visualizing the LCS Algorithm

L	-1	0	1	2	3	4	5	6	7	8	9	10	11
-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	1	1	1	1
1	0	0	1	1	2	2	2	2	2	2	2	2	2
2	0	0	1	1	2	2	2	3	3	3	3	3	3
3	0	1	1	1	2	2	2	3	3	3	3	3	3
4	0	**1**	1	1	2	2	2	3	3	3	3	3	3
5	0	1	1	1	**2**	2	2	3	4	4	4	4	4
6	0	1	1	2	2	**3**	3	3	4	4	5	5	5
7	0	1	1	2	2	3	**4**	4	4	4	5	5	6
8	0	1	1	2	3	3	4	5	**5**	5	5	5	6
9	0	1	1	2	3	4	4	5	5	5	6	6	**6**



Analysis of LCS Algorithm

- We have two nested loops
  - The outer one iterates n times
  - The inner one iterates m times
  - A constant amount of work is done inside each iteration of the inner loop
  - Thus, the total running time is $O(nm)$
- Answer is contained in L[n,m] (and the subsequence can be recovered from the L table).

Exercise

Given two strings, output the LCS

- Example:
  - Inputs: “Fang Yu” and “Shannon Yu”
  - Output: “an Yu”

Hint

```
for i =1 to n-1 do
    L[i,-1] = NULL;
for j =0 to m-1 do
    L[-1,j] = NULL;
for i =0 to n-1 do
    for j =0 to m-1 do
        if x_i = y_j then
            L[i, j] = L[i-1, j-1]+ xi;
        else
            L[i, j] = (L[i-1, j].size() <= L[i, j-1].size())?L[i,j-1]:L[i-1,j];

return L[n-1,m-1] ;
```

HW9 (Due on Dec. 19)

Find the most similar keyword!

- Implement the LCS algorithm for keywords
- Add each keyword into an array/linked list
- Given a string *s*, output the keyword *k*, such that *k*'s value and *s* have the longest common sequence among all the added keywords.

Operations

Given a sequence of operations in a txt file, parse the txt file and execute each operation accordingly

operations	description
add(Keyword k)	Insert a keyword k to an array
find(String s)	Find and output the most similar keyword by using the LCS algorithm

An input file

Similar to HW8,

1. You need to read the sequence of operations from a txt file
2. The format is firm
3. Raise an exception if the input does not match the format

```
add Fang 3
add Yu 5
add NCCU 2
add UCSB 1
add Management 4
add Information 5
find NTU
find Manager
```

NTU: [NCCU, 2]
Manager: [Management, 4]

Coming Up…

- More about dynamic programming is on Ch12
- We will talk about binary Search Tree on Dec. 19.
- Read Text book Ch10

Midterm on Dec. 5 (9:10-12:00am, 大勇樓106)

- Lec 1-10, TextBook Ch1-8, 11-12 (part)
- How to prepare your midterm:
  - Understand "ALL" the materials mentioned in the slides
    - Discuss with me, your TAs, or classmates
    - Read the text book to help you understand the materials
- You are allowed to bring an A4 size note
  - Prepare your own note; write whatever you think that may help you get better scores in the midterm

The MIS Week

- Dec. 9-13. 1F Commerce Building
- Attend MIS Project Demo on Monday Dec. 9
- Sign up for your attendance

- No class on Dec. 12