Answers to Numerical Problems

CHAPTER 2

2.1a $\begin{bmatrix} 4 & 1 & -1 \\ 3 & -4 & 2 \\ 5 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix}$ 2.1b $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 7 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 24 \\ 25 \end{bmatrix}$

2.2a $\begin{bmatrix} 3 & 5 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ 2.2b $\begin{bmatrix} 3 & 6 & 1 \\ 7 & 1 & -1 \end{bmatrix}$ 2.2c $\begin{bmatrix} 3 & 6 & 1 \\ 7 & 1 & -1 \end{bmatrix}$

2.2d $\begin{bmatrix} -1 & -2 & -7 \\ 1 & -1 & 3 \end{bmatrix}$ 2.2e $\begin{bmatrix} -3 & -5 & -1 \\ -3 & -2 & -1 \end{bmatrix}$ 2.2f $\begin{bmatrix} -1 & 0 & -7 \\ 9 & -3 & -1 \end{bmatrix}$

2.3a 26 2.3b $\begin{bmatrix} -2 \\ -4 \\ -8 \end{bmatrix}$ 2.3c (5, 15, 20) 2.3d 23 2.3e 210

2.3f 23/2

2.4a (−7, 8) 2.4b $\begin{bmatrix} 4 & 6 & 8 \\ -2 & 4 & 0 \end{bmatrix}$ 2.4c $\begin{bmatrix} -4 & 12 \\ 3 & -4 \end{bmatrix}$

2.4d $\begin{bmatrix} 0 & 2 & 0 \\ 8 & -2 & -4 \end{bmatrix}$ 2.4e $\begin{bmatrix} 240 & -80 & -120 \\ -80 & 35 & 40 \end{bmatrix}$ 2.4f $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 4 & 12 & 16 \end{bmatrix}$

2.5a

(i) $(\mathbf{DE})' = \mathbf{E}'\mathbf{D}' = \begin{bmatrix} ea + gb & ec + gd \\ fa + hb & fc + hd \end{bmatrix}$

(ii) $\mathbf{D}'\mathbf{E}' = \begin{bmatrix} ae + cf & ag + ch \\ be + df & bg + dh \end{bmatrix}$ (iii) $\mathbf{E}'\mathbf{D}'$ = See (i) above.

2.5b

(i) $(\mathbf{DE})' = \mathbf{E}'\mathbf{D}' = \begin{bmatrix} 3 & 0 \\ 10 & 4 \end{bmatrix}$ (ii) $\mathbf{D}'\mathbf{E}' = \begin{bmatrix} 3 & 0 \\ 17 & 4 \end{bmatrix}$

(iii) $\mathbf{E}'\mathbf{D}'$ = See (i) above.

2.6a $\mathbf{F} \neq \phi$; $\mathbf{G} \neq \phi$; $\mathbf{FG} = \phi$ 2.6b

$$\mathbf{GF} = \begin{bmatrix} -10 & 30 & 50 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \phi$$

2.7a 84 2.7b −30 2.7c 0 2.7d 0

2.8a (−2, 6, 10) 2.8b (−3, 0, 18) 2.8c 28 2.8d (0, 0, 0)

2.9

$$\mathbf{X}^2 - \mathbf{X} - 2\mathbf{I} = (\mathbf{X} + \mathbf{I})(\mathbf{X} - 2\mathbf{I}) = \begin{bmatrix} a^2 + bc - a - 2 & ab + bd - b \\ ac + cd - c & bc + d^2 - d - 2 \end{bmatrix}$$

2.10a $\begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}$ 2.10b $\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$ 2.10c $\begin{bmatrix} 28 & 66 \\ 44 & 105 \end{bmatrix}$ 2.10d $\begin{bmatrix} 30 & 48 \\ 72 & 114 \end{bmatrix}$

2.11a 0 2.11b 0 2.11c 0 2.11d $a^2 + b^2$

2.12a 148 2.12b −32 2.12c −128

2.13 $|\mathbf{A}| = 3(10) + 2 - 17 = 15$

2.14 $|\mathbf{M}| = (1)(-8)(-2.75)(-5.8182) = -128$

2.15a $\Sigma Y = 33$ $\bar{X}_1 = 4.33$ $\Sigma YX_2 = 165$ $\Sigma X_3{}^2 - (\Sigma X_3)^2/m = 50.83$

2.15b

$$\mathbf{S} = \begin{bmatrix} 41.5 & 31 & 38 & -37.5 \\ 31 & 39.3 & 33.3 & -43.7 \\ 38.5 & 33.3 & 38.3 & -38.2 \\ -37.5 & -43.7 & -38.2 & 50.8 \end{bmatrix}$$

2.15c

$$\mathbf{C} = \begin{bmatrix} 6.9 & 5.2 & 6.3 & -6.3 \\ 5.2 & 6.6 & 5.6 & -7.3 \\ 6.3 & 5.6 & 6.4 & -6.4 \\ -6.3 & -7.3 & -6.4 & 8.5 \end{bmatrix}$$

2.15d

$$\mathbf{R} = \begin{bmatrix} 1 & 0.76 & 0.96 & -0.82 \\ 0.76 & 1 & 0.85 & -0.98 \\ 0.96 & 0.85 & 1 & -0.86 \\ -0.82 & -0.98 & -0.86 & 1 \end{bmatrix}$$

2.15e

$$\mathbf{A}_d = \begin{bmatrix} -3.5 & -3.3 & -3.8 & 4.2 \\ -1.5 & -2.3 & -0.8 & 3.2 \\ -2.5 & 0.7 & -1.8 & -0.8 \\ 1.5 & -1.3 & 0.2 & 0.2 \\ 2.5 & 2.7 & 3.2 & -2.8 \\ 3.5 & 3.7 & 3.2 & -3.8 \end{bmatrix}$$

2.15f $-3.5 - 1.5 - 2.5 + 1.5 + 2.5 + 3.5 = 0$

CHAPTER 3

3.1a $\|a\| = \sqrt{5}$; $\|b\| = \sqrt{3}$; $\|c\| = \sqrt{\pi^2 + 7}$

3.1b the plane $y = 1 - x$ for all z

3.1c a parabola $z = x^2$ for all y

3.1d a sphere with center (0, 0, 0) and radius 1

3.2a $\begin{bmatrix} 5/2 \\ 3/2 \\ 1/2 \end{bmatrix}$ 3.2c $\|PR\| = 3\sqrt{3}/2$; $\|RQ\| = 3\sqrt{3}/2$; $\|PQ\| = 3\sqrt{3}$

3.3a $k = -1$ 3.3b $k_1 = 1$; $k_2 = 2$; $k_3 = -1$ 3.3c linearly independent vectors; $k_1 = k_2 = 0$

3.4a 1.414 3.4b 0 3.4c $-\frac{1}{2}$ 3.4d $-6 \leqslant a'b \leqslant 6$

3.5 $\dfrac{|a'c|}{\|c\|} + \dfrac{|b'c|}{\|c\|} = \|a_p\| + \|b_p\|$ (if a′c and b′c have same sign)

3.6

			α	β	γ
3.6a	$a' + b'$	$3\sqrt{10}$	$-1/3\sqrt{10}$	$5/3\sqrt{10}$	$8/3\sqrt{10}$
3.6b	$a' - b'$	$\sqrt{110}$	$5/\sqrt{110}$	$-7/\sqrt{110}$	$6/\sqrt{110}$
3.6c	$5a' + 10b'$	$\sqrt{5450}$	$-20/\sqrt{5450}$	$55/\sqrt{5450}$	$45/\sqrt{5450}$
3.6d	$\frac{1}{2}(a' - b')$	$\sqrt{110}/2$	$5/\sqrt{110}$	$-7/\sqrt{110}$	$6/\sqrt{110}$

$\cos\theta_{xy} = \dfrac{-43}{2\sqrt{5995}}$; where $x' = 5a' + 10b'$; $y' = \frac{1}{2}(a' - b')$

3.7a $a^* = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$ $b^* = \begin{bmatrix} 33/\sqrt{1414} \\ -18/\sqrt{1414} \\ 1/\sqrt{1414} \end{bmatrix}$ $c^* = \begin{bmatrix} 4/\sqrt{101} \\ 7/\sqrt{101} \\ -6/\sqrt{101} \end{bmatrix}$

3.7b $a^* = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ $b^* = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ $c^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

3.7c The third vector vanishes; in a two-dimensional space no more than two vectors can constitute a basis.

3.8 $a^* = \begin{bmatrix} 1/8 \\ 5/8 \end{bmatrix}$

3.9 $a'b = 0$. The orthogonal vector c is of the form $\begin{bmatrix} -2k \\ k \\ k \end{bmatrix}$

3.10 The equation for the circle is independent of parameter Ψ, denoting the angle of rotation. New equations for the ellipse with coordinates (u, v) instead of (x, y) are

3.10a $\frac{5}{2}u^2 + \frac{5}{2}v^2 - 3uv = 4$ **3.10b** $\frac{7}{4}u^2 + \frac{13}{4}v^2 - \frac{3\sqrt{3}}{2}uv = 4$

3.10c $\frac{7}{4}u^2 + \frac{13}{4}v^2 + \frac{3\sqrt{3}}{2}uv = 4$

3.11 The vectors $(4, -2, 1, 7)$ and $(2, -3, x, y)$ are linearly independent for any x, y since $(4, -2)$ and $(2, -3)$ are linearly independent in a two-dimensional space.

3.12
$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \frac{1}{17}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)\begin{bmatrix} 7 & -1 & 3 \\ -8 & 6 & -1 \\ 6 & -13 & 5 \end{bmatrix}$$

3.13a $\mathbf{a}'\mathbf{b} = 7; \quad \mathbf{a}^{*\prime}\mathbf{b}^* = \dfrac{3\sqrt{2} - 11\sqrt{6}}{4}$

3.13b $\|\mathbf{a}\| = \sqrt{5}; \quad \|\mathbf{b}\| = \sqrt{13}; \quad \|\mathbf{a}^*\| = \sqrt{5}; \quad \|\mathbf{b}^*\| = \sqrt{13}$

Vector lengths are preserved under rotation.

3.15 $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 14 & 24 & 18 \\ 6 & 10 & 18 & 14 \end{bmatrix}$

Area of $(ABCD)$ is 4 and area of $(A'B'C'D')$ is 8.

3.16c $\mathbf{y}'\mathbf{x}_2 = m \operatorname{cov}(Y, X_2); \quad \|\mathbf{y}\| = \sqrt{m}\, s_y; \quad \|\mathbf{x}_2\| = \sqrt{m}\, s_x$

CHAPTER 4

4.1a an identity mapping

4.1b a projection on z, followed by a reversal of direction along z

4.1c a reversal of direction along z

4.1d leaves terminus of vector at same height but moves vector three times as far from z.

4.2a $\mathbf{x}_1^* = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}; \quad \mathbf{x}_2^* = \begin{bmatrix} 4 \\ 4 \end{bmatrix}; \quad \mathbf{x}_3^* = \begin{bmatrix} 5 \\ 6\frac{1}{2} \end{bmatrix}$

4.2b $\mathbf{x}_1^* = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}; \quad \mathbf{x}_2^* = \begin{bmatrix} 4 \\ 4 \end{bmatrix}; \quad \mathbf{x}_3^* = \begin{bmatrix} 5 \\ 6\frac{1}{2} \end{bmatrix}; \quad \mathbf{x}_4^* = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

4.2c $\mathbf{x}_5^* = \begin{bmatrix} 1 \\ 5/2 \end{bmatrix}; \quad \mathbf{x}_6^* = \begin{bmatrix} 2 \\ 3 \end{bmatrix}; \quad \mathbf{x}_7^* = \begin{bmatrix} 2 \\ 6 \end{bmatrix}; \quad \mathbf{x}_8^* = \begin{bmatrix} 1 \\ 5\frac{1}{2} \end{bmatrix}$

Areas are invariant while angles and lengths change.

4.3a (i) $\begin{bmatrix} 2 & 3 \\ 8 & 6 \\ 10 & 12 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 \\ 8 & 6 \\ 10 & 12 \\ 4 & 9 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 6 \\ 4 & 6 \\ 4 & 15 \\ 2 & 15 \end{bmatrix}$

4.3b (i) $\begin{bmatrix} 3 & 1 \\ 8 & 2 \\ 13 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 \\ 8 & 2 \\ 13 & 4 \\ 8 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 5 & 2 \\ 6 & 2 \\ 12 & 5 \\ 11 & 5 \end{bmatrix}$

4.4a $(x_1^*, x_2^*) = \left\{(x_1, x_2) \begin{bmatrix} 0.707 & 0.707 \\ -0.707 & 0.707 \end{bmatrix}\right\} + (2, -1)$

4.4b $(x_1^*, x_2^*) = (x_1, x_2)\begin{bmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1/2 \end{bmatrix}$

4.4c $(x_1^*, x_2^*) = (x_1, x_2)\begin{bmatrix} 3 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{bmatrix}$

4.5a $\begin{bmatrix} 1/2 & 1 \\ 0 & 1/2 \end{bmatrix}$ **4.5b** $\begin{bmatrix} 0.35 & 1.06 \\ -0.35 & -0.35 \end{bmatrix}$

4.5c $\begin{bmatrix} 0.35 & 0.35 \\ -0.35 & 0.35 \end{bmatrix}$ **4.5d** $\begin{bmatrix} 0.35 & 1.06 \\ -0.35 & -0.35 \end{bmatrix}$

4.6a $\mathbf{A}^{-1} = \begin{bmatrix} 5/11 & -2/11 \\ -7/11 & 5/11 \end{bmatrix}$ **4.6b** $\mathbf{A}^{-1}$ is not defined

4.6c $\mathbf{A}^{-1}$ is not defined. **4.6d** $\mathbf{A}^{-1} = \dfrac{-1}{ad + bc}\begin{bmatrix} -d & b \\ c & a \end{bmatrix}$

4.7a 2 **4.7b** 3 **4.7c** 3

4.8a $\begin{bmatrix} 1 & 2 & 3 & 9 & 2 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & -17/2 & -7/2 \\ 0 & 0 & 0 & 1 & -8/37 \end{bmatrix}$ **4.8b** $\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ **4.8c** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4.9a $r(\mathbf{A}) = 2$; $r(\mathbf{B}) = 3$. Regular inverse does not exist.

4.9b $r(\mathbf{A}) = 3 = r(\mathbf{B}) = 3$; $\mathbf{A}^{-1} = \begin{bmatrix} -1/3 & -2 & 20/6 \\ 1/6 & 1 & -7/6 \\ 1/6 & 0 & -1/6 \end{bmatrix}$

4.9c $r(\mathbf{A}) = r(\mathbf{B}) = 2$. Regular (3×3) inverse does not exist.

4.10 $\mathbf{A}^{-1} = 1/4 \begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$; $|\mathbf{A}| = 4$; $x_1^* = -1/2$; $x_2^* = -1/2$; $x_3^* = 3/2$

4.11 $\mathbf{R}^{-1} = \begin{bmatrix} 10.26 & -9.74 \\ -9.74 & 10.26 \end{bmatrix}$; $|\mathbf{R}| = 0.0975$; $\mathbf{b}^* = \begin{bmatrix} 1.072 \\ -0.128 \end{bmatrix}$

4.12a $\mathbf{x}^{\circ *} = \begin{bmatrix} -377 \\ 1406 \end{bmatrix}$ 4.12b $\mathbf{x} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$ 4.12c $|\mathbf{T}| = -2$

$|\mathbf{T}^{\circ}| = -2$

4.13
$$\begin{array}{ccc} \mathbf{x}^{\circ *} & \mathbf{T}^{\circ} & \mathbf{x}^{\circ} \\ \begin{bmatrix} x_1{}^{\circ *} \\ x_2{}^{\circ *} \\ x_3{}^{\circ *} \end{bmatrix} = & \begin{bmatrix} 7 & 1 & 10 \\ 0 & 1 & 0 \\ -4 & 0 & -6 \end{bmatrix} & \begin{bmatrix} x_1{}^{\circ} \\ x_2{}^{\circ} \\ x_3{}^{\circ} \end{bmatrix} \end{array}$$

4.14
$$\begin{array}{ccc} \mathbf{x}^{\circ *} & \mathbf{T}^{\circ} & \mathbf{x}^{\circ} \\ \begin{bmatrix} x_1{}^{\circ *} \\ x_2{}^{\circ *} \\ x_3{}^{\circ *} \end{bmatrix} = & \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} & \begin{bmatrix} x_1{}^{\circ} \\ x_2{}^{\circ} \\ x_3{}^{\circ} \end{bmatrix} \end{array}$$

$\mathbf{T}^{\circ}$ is a stretch transformation.

CHAPTER 5

5.1a $\lambda_1 = 5$; $\lambda_2 = 8$; $\mathbf{U} = \begin{bmatrix} 3/\sqrt{58} & 0 \\ -7/\sqrt{58} & 1 \end{bmatrix}$

5.1b $\lambda_1 = -4$; $\lambda_2 = -6$; $\mathbf{U} = \begin{bmatrix} 2/\sqrt{5} & 4/\sqrt{17} \\ 1/\sqrt{5} & 1/\sqrt{17} \end{bmatrix}$

5.1c $\lambda_1 = 2$; $\lambda_2 = -7$; $\mathbf{U} = \begin{bmatrix} \sqrt{2}/2 & -5/\sqrt{41} \\ \sqrt{2}/2 & 4/\sqrt{41} \end{bmatrix}$

5.1d $\lambda_1 = 5$; $\lambda_2 = 4$; $\mathbf{U} = \begin{bmatrix} 1/\sqrt{10} & 1/\sqrt{5} \\ -3/\sqrt{10} & -2/\sqrt{5} \end{bmatrix}$

5.1e $\lambda_1 = 4$; $\lambda_2 = 2$; $\lambda_3 = -2$;
$$\mathbf{U} = \begin{bmatrix} 1/\sqrt{11} & 3/\sqrt{14} & \sqrt{2}/2 \\ 3/\sqrt{11} & 2/\sqrt{14} & 0 \\ 1/\sqrt{11} & 1/\sqrt{14} & \sqrt{2}/2 \end{bmatrix}$$

5.1f $\lambda_1 = 3$; $\lambda_2 = 3$; $\lambda_3 = 3$;
$$\mathbf{U} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

All eigenvectors are of the form $\begin{bmatrix} k \\ k \\ k \end{bmatrix}$

5.2a $\mathrm{tr}(\mathbf{A}) = 13$; $|\mathbf{A}| = 40$ 5.2b $\mathrm{tr}(\mathbf{A}) = -10$; $|\mathbf{A}| = 24$

5.2c $\mathrm{tr}(\mathbf{A}) = -5$; $|\mathbf{A}| = -14$ 5.2d $\mathrm{tr}(\mathbf{A}) = 9$; $|\mathbf{A}| = 20$

5.3a $\begin{bmatrix} 0 \\ k \end{bmatrix}$ 5.3b $\begin{bmatrix} k \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ k \end{bmatrix}$ 5.3c Any vector remains invariant under central dilation.

5.3d There is no vector remaining invariant under this transformation since eigenvalues are complex.

5.4 The matrix $\mathbf{A} = \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix}$ has eigenvalues given by the diagonal elements of $\mathbf{D} = \begin{bmatrix} 4+\sqrt{2} & 0 \\ 0 & 4-\sqrt{2} \end{bmatrix}$ and associated eigenvectors that are

$$\mathbf{u}_1 = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}; \qquad \mathbf{u}_2 = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}$$

as columns of $\mathbf{U}$. Then, let $\mathbf{A} = \mathbf{UDU}'$.

5.4a $\mathbf{U}\begin{bmatrix} 12+3\sqrt{2} & 0 \\ 0 & 12-3\sqrt{2} \end{bmatrix}\mathbf{U}'$ 5.4b $\mathbf{U}\begin{bmatrix} 6+\sqrt{2} & 0 \\ 0 & 6-\sqrt{2} \end{bmatrix}\mathbf{U}'$

5.4c $\mathbf{U}\begin{bmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{bmatrix}\mathbf{U}'$ 5.4d $\mathbf{U}\begin{bmatrix} 88+50\sqrt{2} & 0 \\ 0 & 88-50\sqrt{2} \end{bmatrix}\mathbf{U}'$

5.4e $\mathbf{U}\begin{bmatrix} \dfrac{1}{4+\sqrt{2}} & 0 \\ 0 & \dfrac{1}{4-\sqrt{2}} \end{bmatrix}\mathbf{U}'$ 5.4f $\mathbf{U}\begin{bmatrix} \sqrt{4+\sqrt{2}} & 0 \\ 0 & \sqrt{4-\sqrt{2}} \end{bmatrix}\mathbf{U}'$

5.4g $\mathbf{U}\begin{bmatrix} (4+\sqrt{2})^{-1/2} & 0 \\ 0 & (4-\sqrt{2})^{-1/2} \end{bmatrix}\mathbf{U}'$

5.5 $$\mathbf{A}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}; \qquad \mathbf{ABA}^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5.6 $\lambda_1 = 12$; $\lambda_2 = 6$; $\lambda_3 = 6$. The last two eigenvalues are not unique. One possible solution is

$$\mathbf{U} = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}$$

Then

$$\mathbf{D} = \mathbf{U}'\mathbf{AU} = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

5.7a 3 5.7b 3 5.7c 2 5.7d 1

5.8a $$\mathbf{A} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

5.8b

$$\mathbf{A} = \underset{\mathbf{P}}{\begin{bmatrix} -0.650 & 0.646 \\ -0.616 & -0.173 \\ 0.430 & 0.744 \end{bmatrix}}\underset{\mathbf{\Delta}}{\begin{bmatrix} 2.022 & 0 \\ 0 & 3.861 \end{bmatrix}}\underset{\mathbf{Q}'}{\begin{bmatrix} 0.290 & -0.957 \\ 0.957 & 0.290 \end{bmatrix}}$$

5.8c
$$\mathbf{A} = \begin{bmatrix} 0.707 & 0.707 \\ -0.707 & 0.707 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5.8d
$$\mathbf{A} = \begin{bmatrix} 0.114 & 0.634 \\ 0.282 & -0.108 \\ 0.396 & 0.526 \\ 0.553 & -0.551 \\ 0.667 & 0.083 \end{bmatrix} \begin{bmatrix} 10.592 & 0 \\ 0 & 2.969 \end{bmatrix} \begin{bmatrix} 0.994 & -0.107 \\ 0.107 & 0.994 \end{bmatrix}$$

In all four cases, $r(\mathbf{A}) = 2$.

5.9a

(5.8a):
$$(\mathbf{A}'\mathbf{A})^2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

(5.8b):
$$(\mathbf{A}'\mathbf{A})^2 = \begin{bmatrix} 0.290 & 0.957 \\ -0.957 & 0.290 \end{bmatrix} \begin{bmatrix} 16.744 & 0 \\ 0 & 227.248 \end{bmatrix} \begin{bmatrix} 0.290 & -0.957 \\ 0.957 & 0.290 \end{bmatrix}$$

5.9b

(5.8a):
$$(\mathbf{A}'\mathbf{A})^{1/2} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

(5.8b):
$$(\mathbf{A}'\mathbf{A})^{1/2} = \begin{bmatrix} 0.290 & 0.957 \\ -0.957 & 0.290 \end{bmatrix} \begin{bmatrix} 2.022 & 0 \\ 0 & 3.861 \end{bmatrix} \begin{bmatrix} 0.290 & -0.957 \\ 0.957 & 0.290 \end{bmatrix}$$

5.10a
$$\mathbf{A}^{1/2} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{7} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{A}^{-1/2} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{7} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

5.10b
$$\mathbf{A}^{1/2} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{A}^{-1/2} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 0 \\ 0 & 1/2\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

5.10c
$$\mathbf{A}^{1/2} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$\mathbf{A}^{-1/2} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

5.10d
$\mathbf{A} = \mathbf{U}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{U}'$, where $\mathbf{U}$ is any set of two orthonormal vectors.

$$\mathbf{A}^{1/2} = \mathbf{U}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\mathbf{U}'; \qquad \mathbf{A}^{-1/2} = \mathbf{U}\begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}\mathbf{U}'$$

5.11a $\mathbf{x}'\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\mathbf{x}$; rank 1

5.11b $\mathbf{x}'\begin{bmatrix}1 & -2\\ -2 & 2\end{bmatrix}\mathbf{x}$; rank 2

5.11c $\mathbf{x}'\begin{bmatrix}9 & -3\\ -3 & 1\end{bmatrix}\mathbf{x}$; rank 1

5.11d $\mathbf{x}'\begin{bmatrix}2 & -3/2\\ -3/2 & 3\end{bmatrix}\mathbf{x}$; rank 2

5.12a $\mathbf{A} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$

Rotation–projection on first axis and stretch-rotation

5.12b $\mathbf{A} = \mathbf{U}\begin{bmatrix}\dfrac{3+\sqrt{17}}{2} & 0\\ 0 & \dfrac{3-\sqrt{17}}{2}\end{bmatrix}\mathbf{U}'$

Rotation–stretch–rotation

5.12c $\mathbf{A} = \begin{bmatrix}-3/\sqrt{10} & 1/\sqrt{10}\\ 1/\sqrt{10} & 3/\sqrt{10}\end{bmatrix}\begin{bmatrix}10 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-3/\sqrt{10} & 1/\sqrt{10}\\ 1/\sqrt{10} & 3/\sqrt{10}\end{bmatrix}$

Rotation–projection on first axis and stretch-rotation

5.12d $\mathbf{A} = \mathbf{U}\begin{bmatrix}\dfrac{5+\sqrt{10}}{2} & 0\\ 0 & \dfrac{5-\sqrt{10}}{2}\end{bmatrix}\mathbf{U}'$

Rotation–stretch–rotation

5.13a $\mathbf{C} = \begin{bmatrix}29.52 & 19.44 & 14.44\\ 19.44 & 14.19 & 10.69\\ 14.44 & 10.69 & 8.91\end{bmatrix}$

5.13b
$$Z_1 = 0.755Y_d + 0.521X_{d1} + 0.396X_{d2};\quad \lambda_1 = 50.50$$
$$Z_2 = 0.618Y_d - 0.366X_{d1} - 0.696X_{d2};\quad \lambda_2 = 1.72$$
$$Z_3 = 0.218Y_d - 0.771X_{d1} + 0.598X_{d2};\quad \lambda_3 = 0.39$$

5.14a $\mathbf{A} = \mathbf{x}'\begin{bmatrix}-3/\sqrt{10} & 1/\sqrt{10}\\ 1/\sqrt{10} & 3/\sqrt{10}\end{bmatrix}\begin{bmatrix}10 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}-3/\sqrt{10} & 1/\sqrt{10}\\ 1/\sqrt{10} & 3/\sqrt{10}\end{bmatrix}\mathbf{x}$

Positive semidefinite

5.14b $\mathbf{x}'\begin{bmatrix}0.542 & -0.707 & 0.455\\ -0.643 & 0 & 0.765\\ 0.542 & 0.707 & 0.455\end{bmatrix}\begin{bmatrix}-7.372 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -1.628\end{bmatrix}\begin{bmatrix}0.542 & -0.643 & 0.542\\ -0.707 & 0 & 0.707\\ 0.455 & 0.765 & 0.455\end{bmatrix}\mathbf{x}$

Negative definite

5.14c $\mathbf{x}'\begin{bmatrix}0.585 & 0.811\\ -0.811 & 0.585\end{bmatrix}\begin{bmatrix}4.081 & 0\\ 0 & 0.919\end{bmatrix}\begin{bmatrix}0.585 & -0.811\\ 0.811 & 0.585\end{bmatrix}\mathbf{x}$

Positive definite

5.14d $\mathbf{x}'\begin{bmatrix}0.707 & -0.408 & -0.577\\ 0.707 & 0.408 & 0.577\\ 0 & 0.816 & -0.577\end{bmatrix}\begin{bmatrix}6 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0.707 & 0.707 & 0\\ -0.408 & 0.408 & 0.816\\ -0.577 & 0.577 & -0.577\end{bmatrix}\mathbf{x}$

Positive semidefinite

5.15a

$$\mathbf{W}^{-1/2} = \begin{bmatrix} 0.393 & -0.0424 \\ -0.0424 & 0.345 \end{bmatrix}$$

$$\mathbf{W}^{-1/2}\mathbf{A}\mathbf{W}^{-1/2} = \begin{bmatrix} 21.216 & 13.203 \\ 13.203 & 8.257 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0.849 & -0.529 \\ 0.529 & 0.849 \end{bmatrix}}_{\mathbf{Q}_2} \underbrace{\begin{bmatrix} 29.444 & 0 \\ 0 & 0.029 \end{bmatrix}}_{\mathbf{\Lambda}} \underbrace{\begin{bmatrix} 0.849 & 0.529 \\ -0.529 & 0.849 \end{bmatrix}}_{\mathbf{Q}_2'}$$

$$\mathbf{Y} = \mathbf{X}_d \mathbf{W}^{-1/2}\mathbf{Q}_2 = \mathbf{X}_d \begin{bmatrix} 0.311 & -0.244 \\ 0.146 & 0.315 \end{bmatrix}$$

5.15b Both transformations yield same diagonal matrix.

CHAPTER 6

6.1a $\hat{Y} = -2.263 + 1.550X_1 - 0.239X_2$ $\quad \mathbf{C} = \begin{bmatrix} 14.19 & 10.69 \\ 10.69 & 8.91 \end{bmatrix}$

6.1b $\hat{Y} = -2.263 + 1.550X_1 - 0.239X_2$ $\quad \mathbf{R} = \begin{bmatrix} 1 & 0.951 \\ 0.951 & 1 \end{bmatrix}$

$b_1^* = 1.076$; $b_2^* = -1.309$

6.1c $\hat{Y} = 1.125 - 0.179X_1 + 0.117X_1^2$

$\mathbf{R} = \begin{bmatrix} 1 & 0.978 \\ 0.978 & 1 \end{bmatrix}$; $R^2 = 0.954$

$R^2 = 0.954$ exceeds $R^2 = 0.902$ for Y regressed on X_1 alone.

6.2a $\mathbf{R} = \begin{bmatrix} 1 & 0.951 \\ 0.951 & 1 \end{bmatrix}$; $\quad \mathbf{T} = \begin{bmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{bmatrix}$

$Z_1 = 0.707X_{s1} + 0.707X_{s2}$; $\quad \lambda_1 = 1.951$

$Z_2 = 0.707X_{s1} - 0.707X_{s2}$; $\quad \lambda_2 = 0.049$

$$\mathbf{F} = \underbrace{\begin{bmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} 1.398 & 0 \\ 0 & 0.208 \end{bmatrix}}_{\mathbf{\Lambda}^{1/2}} = \begin{bmatrix} 0.988 & 0.147 \\ 0.988 & -0.147 \end{bmatrix}$$

A 45° rotation is involved. Eigenvectors (and loadings) differ from those obtained from **C**, the covariance matrix.

6.2b $\mathbf{A} = \begin{bmatrix} 53.25 & 41.42 \\ 41.42 & 33.08 \end{bmatrix}$; $\quad \mathbf{T} = \begin{bmatrix} 0.787 & 0.617 \\ 0.617 & -0.787 \end{bmatrix}$

$\lambda_1 = 85.79$; $\quad \lambda_2 = 0.54$. $\quad$ Rotation is 38°.

6.2c R^2 is the same if both columns of component scores are used. R^2 is lower ($R^2 = 0.88$) if only the first column of component scores is used.

6.2d $\mathbf{C} = \begin{bmatrix} 29.52 & 19.44 & 14.44 \\ 19.44 & 14.19 & 10.69 \\ 14.44 & 10.69 & 8.91 \end{bmatrix}$ $\quad \mathbf{T} = \begin{bmatrix} 0.756 & -0.618 & -0.218 \\ 0.521 & 0.366 & 0.771 \\ 0.396 & 0.696 & 0.599 \end{bmatrix}$

$\lambda_1 = 50.51; \quad \lambda_2 = 1.72; \quad \lambda_3 = 0.398$

6.3a $\mathbf{V}_s = \begin{bmatrix} 0.937 & -0.700 \\ 0.348 & 0.715 \end{bmatrix}; \quad \mathbf{\Lambda} = \begin{bmatrix} 29.155 & 0 \\ 0 & 0.028 \end{bmatrix}$

Discriminant weights change under standardization of data matrix.

6.3b $\mathbf{W}^{-1/2}\mathbf{Q}_2 = \begin{bmatrix} 0.905 & -0.612 \\ 0.425 & 0.791 \end{bmatrix}$

As can be seen from Table 6.5, $\mathbf{W}^{-1/2}\mathbf{Q}_2$ is equal to $\mathbf{V}$, the matrix of eigenvectors obtained from $\mathbf{W}^{-1}\mathbf{A}$.

6.3c $\lambda_1 = 3.768; \quad \mathbf{t}_1 = \begin{bmatrix} 0.868 \\ -0.497 \end{bmatrix}$

In the two-group case only one discriminant is computed.

6.4a $\mathbf{C} = \begin{bmatrix} 14.19 & 1.46 \\ 1.46 & 0.25 \end{bmatrix}; \quad \mathbf{R} = \begin{bmatrix} 1 & 0.774 \\ 0.774 & 1 \end{bmatrix}$

$\hat{Y} = -2.488 + 1.496X_1 - 1.229X_2$

6.4b $R^2 = 0.907$

APPENDIX A

A.1a $\dfrac{2-2x}{2x-x^2}$ **A.1b** $\dfrac{2(x^2-x-4)}{(x^2+4)^2}$ **A.1c** $\dfrac{1}{2\sqrt{x}}\, e^{x^{1/2}}$ **A.1d** $\dfrac{4\,(x^3-1)^3\,(9x^2-12x^5)}{(2x^3+1)^5}$

A.2 $x = -1$ (minimum); $\quad x = 1$ (maximum)

A.3 $\dfrac{\partial f}{\partial x}(1, 3) = 4.2; \quad \dfrac{\partial f}{\partial y}(1, 3) = 4.6$ **A.4** $\min f(x, y) = -18$

A.5 $f(-2/3, 4/3) = 16/3$ (maximum)

A.6 $\dfrac{\partial g}{\partial \mathbf{x}}(\mathbf{x}) = 2\mathbf{A}x$. If $\mathbf{x}' = (3, 1, 2)$, $\dfrac{\partial g}{\partial \mathbf{x}} = \begin{bmatrix} 34 \\ 28 \\ 40 \end{bmatrix}$

A.7 $\mathbf{x} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ (minimum)

APPENDIX B

B.1a $r(\mathbf{A}) = 1$ **B.1b** $r(\mathbf{B}) = 2$ **B.1c** $r(\mathbf{C}) = 2$ **B.1d** $r(\mathbf{D}) = 2$

B.2a $\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ **B.2b** $\left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right]$

B.3a $\mathbf{A}^{-1} = \begin{bmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{bmatrix}$ **B.3b** $\mathbf{A}^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

B.4

$$\mathbf{A}^{+} = \begin{bmatrix} 2.56 & -2.11 & -1.11 & 2.34 \\ -1.37 & 1.40 & 0.41 & -1.22 \\ -0.78 & 0.55 & 0.55 & -0.67 \end{bmatrix}$$

B.5

$$\mathbf{x} = \begin{bmatrix} -3\gamma_2 + (1/5)\gamma_5 + 6/5 \\ \gamma_2 \\ 3/5(\gamma_5 + 1) \\ \gamma_5 - 1 \\ \gamma_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

B.6a $\mathbf{A}^{+} = \begin{bmatrix} -0.253 & 0.226 & 0.266 \\ 0.412 & -0.107 & -0.067 \end{bmatrix}$ **B.6b** $\mathbf{A}^{-} = \begin{bmatrix} -0.2 & 0.6 & 0 \\ 0.4 & -0.2 & 0 \end{bmatrix}$