Pricing Black-Scholes Options with Correlated Interest Rate Risk and Credit Risk: An Extension

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Abstract

This article provides a closed-form valuation formula for the Black-Scholes options subject to interest rate risk and credit risk. Not only does our model allow for the possible default of the option issuer prior to the option’s maturity, but also considers the correlations among the option issuer’s total asset, the underlying stock, and the default-free zero coupon bond. We further tailor-make a specific credit-linked option for hedging the default risk of the option issuer. The numerical results show that the default risk of the option issuer significantly reduces the option values, and the vulnerable option values may be remarkably overestimated in the case where the default can occur only at the maturity of the option.

Keywords: Vulnerable Options, Credit Risk, Interest Rate Risk, Time-Changed Brownian Motion
1. Introduction

The traditional Black-Scholes option pricing formula is derived under the assumption that there is no default risk of the option issuer. For exchange-traded options, this is reasonable since most exchanges have been very successful in organizing themselves to ensure that their contracts are always honored. For the over-the-counter (OTC) contracts, on the other hand, the counterparty default risk is important. In recent years, OTC options have become increasingly popular, and hence the default risk of the option issuer should be considered in the pricing of the OTC options.

A vulnerable option is the option that the issuer may default. Generally speaking, there are two pricing methods for modeling credit risk: the reduced-form approach and the structural approach. Hull and White (1995) construct a general reduced-form model and analyze the effects of credit risk on European and American options, which allows for the possible default of the option issuer prior to the maturity of the option. The analytic formula is derived in the case that the underlying asset of the option is assumed to be independent of the option issuer’s asset. Jarrow and Turnbull (1995) use a foreign currency analogy together with a pre-specified default process which is also independent of the underlying asset of the defaultable claims, and obtain the valuation of defaultable claims in both the discrete-time and continuous-time frameworks. However, the independence assumption, used to keep the pricing formulas reasonably tractable, seems only sensible for the case that the asset of the option writer is well-diversified or fully-hedged.

The structural approach for pricing European vulnerable options is first proposed by Johnson and Stulz (1987). They assume the financial distress can occur only at the maturity of the option, and when it is the case, the option holders take over all the assets of the option issuer. Their model also allows for the correlation between the value of the
option issuer’s total asset and that of the underlying stock, which is an important determinant of the vulnerable option prices. Klein (1996) extends Johnson and Stulz (1987) to consider other liabilities in the capital structure of the option issuer. Subsequently, Klein and Inglis (1999) (hereafter, KI (1999)) generalize Klein (1996) with the Vasicek (1977) stochastic interest rate model. Taking the correlations among the underlying stock price, the risk-free interest rate, and the option issuer’s total asset value into consideration, they derive the closed-form pricing formula of the vulnerable option via the partial differential equation (PDE) approach. Recently, Klein and Inglis (2001) employ numerical methods to investigate the generalized default barrier which is the sum of the variable default barrier of Johnson and Stulz (1987) and the fixed default barrier of Klein (1996). All structural models mentioned above suppose that the default of the option issuer only occurs at the maturity of the option, which seems to be a defect of those models.

In this paper, we construct a structural model to value the Black-Scholes vulnerable options, considering not only the possible default of the option issuer prior to the maturity of the option, but also the correlations among the total asset value of the option writer, the value of the underlying stock, and the value of the default-free zero coupon bond. Moreover, we apply the forward risk-neutral pricing approach together with the technique of the time-changed Brownian motion to derive the closed-form valuation of the vulnerable option under stochastic interest rates. Not only does our model provide a framework to overcome the weakness of the previous structural models which the option issuer can default only at the maturity (e.g., Klein (1996) and KI (1999)), but also resolves the limitation of the reduced-form approach which the default of the option issuer and the underlying stock prices are assumed to be independent (e.g., Hull and White (1995) and Jarrow and Turnbull (1995)). The present
paper also tailor-makes a specific credit-linked option to hedge the default risk of the vulnerable option. The numerical results show that the default risk of the option issuer is critical for the valuation of the option, and the vulnerable option prices may be significantly overestimated when the default can occur only at the maturity of the option. Finally, the model can be further extended to allow the value of the option issuer’s total liability to be stochastic, and can also be applied to the valuation of a vulnerable zero-coupon bond option.

The remainder of this paper is organized as follows. In Section 2, we set up the basic assumptions and outline the pricing framework. Section 3 is devoted to the derivation of the closed-form valuation of the Black-Scholes vulnerable options and the corresponding hedging strategy under stochastic interest rates. Next, we present some numerical analysis and explore the properties of the default risk of the options issuer in Section 4. Besides, two applications of our model are provided in Section 5. Finally, Section 6 summarizes the article and makes concluding remarks.

2. The Valuation Framework

In this section, the traditional assumptions of the Black-Scholes economy under stochastic interest rates are employed, and moreover, we extend the economy to allow for the possible default of the option issuer prior to the maturity of the option. The main assumptions of our model are introduced and discussed as follows.

**Assumption 1:** Let \( S(t) \) denote the underlying stock price at time \( t \). The process of \( S(t) \) on the (spot) risk-neutral filtered probability space \( \left( \Omega, \mathcal{F}, \mathcal{Q}, (\mathcal{F}_t)_{t=0}^{T^*} \right) \), where \( 0 \leq t \leq T < T^* \) and \( T^* \) is the termination date of this economy, is given by
\[ \frac{dS(t)}{S(t)} = r(t)dt + \sigma_s(t)dW_s(t), \quad (1) \]

where \( r(t) \) denotes the instantaneous risk-free interest rate, \( \sigma_s(t) \) is the time-varying volatility of the instantaneous rate of return of \( S(t) \), and \( W_s \) is a Wiener process on the same space.

**Assumption 2:** Let \( B(t) \) denote the value of the savings account (or money market account) at time \( t \). If one initially saves \( B(0) \), defined to be one dollar, he/she can obtain the amount of \( B(t) \), which is equal to \( \exp \left( \int_0^t r(s)ds \right) \), at time \( t > 0 \).

**Assumption 3:** Let \( B(t,T) \) denote the time \( t \) value of the default-free zero coupon bond (ZCB) which pays one dollar at time \( T \), for \( 0 \leq t \leq T \). The dynamics of \( B(t,T) \) on the risk-neutral filtered probability space is given by

\[ \frac{dB(t,T)}{B(t,T)} = r(t)dt + b(t,T)dW_b(t), \quad (2) \]

where \( b(t,T) \) is the time-varying volatility of the instantaneous rate of return of \( B(t,T) \) and \( W_b \) is a Wiener process on the same space.

**Assumption 4:** Let \( V(t) \) denote the total asset value of the option issuer at time \( t \).

The evolution of \( V(t) \) on the risk-neutral filtered probability space is given by

\[ \frac{dV(t)}{V(t)} = r(t)dt + \sigma_v(t)dW_v(t), \quad (3) \]

where \( \sigma_v(t) \) is the time-varying volatility of the instantaneous rate of return of \( V(t) \), and \( W_v \) is a Wiener process on the same space. Moreover, the constant instantaneous
correlation coefficients among the above three Wiener processes are given by

$$\rho_{ij} dt \equiv E\left[dW_i(t)dW_j(t)\right], \text{ for } i, j = S, V, B \text{ and } i \neq j,$$

where the expectation is taken under risk neutrality.

**Assumption 5:** Trading takes place continuously in time, and unrestricted borrowing and lending of funds is possible at the same instantaneous risk-free interest rate. Furthermore, the markets of these tradable assets, including $S(t), \ B(t), \ B(t,T), \text{ and } V(t)$, are frictionless, i.e., there are no transaction costs or taxes, and no limitation against the short sales.

**Assumption 6:** There is a pre-specified stochastic default threshold of the option issuer’s total asset value, which is defined as $D \cdot B(t,T)$, where $D$ is a constant representing the expected market value of the options issuer’s total liability at $T$. Once the total asset value of the option issuer goes down and touches this threshold at the time of default $\tau_D$, defined by $\inf\{t < s \leq T : V(s) \leq D \cdot B(s,T)\}$, the option issuer will declare restructuring or bankruptcy.

**Assumption 7:** If restructuring or bankruptcy occurs, the option issuer’s asset is immediately liquidated and the scrap value is $(1-\alpha)V(\tau_D)$, where $\alpha$ is assumed to be a constant showing the ratio of bankruptcy or restructuring costs of the issuer’s asset ($0 \leq \alpha \leq 1$). The recovery rate of the option holder therefore equals $(1-\alpha)V(\tau_D)/D^*$ at $\tau_D$, where $D^* \equiv D \cdot B(\tau_D,T)$ denotes the expected market value of the option issuer’s total liability at $\tau_D$. Upon default, the option holder can directly receive the recovery value (in dollar amount), which is equal to the recovery rate
multiplied by the nominal claim, the market value of the non-vulnerable option on the same underlying asset at $\tau_D$.

Assumptions 1-3 and Assumption 5 are the standard assumptions in the Black-Scholes economy with stochastic interest rates under risk neutrality. Assumption 1 represents the uncertainty source of the underlying stock under risk neutrality. Assumption 2 provides the economy an instantaneously risk-free asset (the savings account). Then, Assumption 3 exhibits the uncertainty source of the price of the default-free ZCB under risk neutrality, which is equivalent to model the uncertainty of interest rates in the same risk-neutral world. Here we do not specify any stochastic interest rate model but assume the stochastic behavior of ZCB prices directly instead. Many stochastic interest rate models, such as Vasicek (1977) and Heath et al. (1992), can be used in our framework by defining the appropriate form of $b(t,T)$. Assumption 4 shows the uncertainty source of the option issuer, which will be utilized to model the possible default of the option issuer, and the correlations among the three uncertainty sources are also defined. Next, Assumption 5 is imposed to ensure that our economy can work ideally as the traditional Black-Scholes environment. Assumption 6 supposes the time when the option issuer will go bankrupt in our model and finally, Assumption 7 expresses the recovery value received by the option holder when the option issuer defaults.

To highlight the setup of the option issuer’s default risk in our framework, we explain Assumptions 6 and 7 in more details. Assumption 6 means that as long as the total asset value of the option issuer falls below the stochastic default threshold $D \cdot B(t,T)$, the bankruptcy or restructuring takes place immediately. In terms of the forward price, it is equivalent to say that once the forward value of the option issuer’s
total asset at time $t$, $F_v(t,T) = V(t)/B(t,T)$, goes down and touches the constant barrier $D$, the option issuer then goes bankrupt. Here $D$ can also be explained as the expected default trigger of the forward value of the issuer’s total asset. As a result, the time of default $\tau_D$ is the first passage time that the value of the issuer’s total asset falls below the stochastic default trigger $D \cdot B(t,T)$, or equivalently, the forward value of the issuer’s total asset goes down and touches the constant default barrier $D$. An important implication of this assumption is that default occurs on all debt obligations simultaneously. This is realistic since when a firm defaults on an obligation, it typically defaults on other obligations as well due to the normal cross-default provision. Notice that at the maturity of the option, the threshold of the issuer’s total asset value is equal to $D$, which is consistent with the fixed default boundary of KI (1999).

In Assumption 7, the recovery rate at the time of default is $(1-\alpha)V(\tau_D)/D^* = 1-\alpha$ (by Assumptions 6 and 7), where $(1-\alpha)V(\tau_D)$ represents the scrap value of the option issuer’s total asset at the time of default. Taking the vulnerable call options as an example, the recovery value (in dollar amount) of the option holder at the time of default is equal to $(1-\alpha)C(\tau_D) = (1-\alpha)B(\tau_D,T)E^{\mathbb{P}_T}[\left((S(T)-K)^+\right)_{\tau_D}]$. By the no-arbitrage argument, the equivalent dollar amount of the option holder’s the recovery value at the maturity is, therefore, equal to $(1-\alpha)E^{\mathbb{P}_T}[\left((S(T)-K)^+\right)_{\tau_D}]$ which depends on the information set at $\tau_D$, i.e., it is $\tau_D$-measurable. The equivalent dollar value is then used to price the vulnerable options in the later section, and the approach adopted here is similar to that of Briys and de Varenne (1997), which investigates the valuation of risky corporate debts.
3. Valuing Vulnerable Options with Stochastic Interest Rates

In this section, we first provide the traditional Black-Scholes formulas under stochastic interest rates. Next, we obtain the pricing formulas for the Black-Scholes vulnerable options where the option issuer can default only at the maturity by our pricing methodology and then make a connection to KI (1999). Subsequently, we derive the pricing formulas for the Black-Scholes vulnerable options where the option issuer can default throughout the remaining life of the option.

For comparison purpose the traditional non-default Black-Scholes pricing formulas under stochastic interest rates are summarized below in Proposition 1.

**Proposition 1:** Denote \( F_s(t,T) = S(t)/B(t,T) \) as the forward price of the underlying stock. The time \( t \) values of the European call option \( C(t) \) and put option \( P(t) \) without the option issuer’s default risk are respectively given by

\[
C(t) = S(t) \Phi(d_1) - KB(t,T) \Phi(d_2),
\]

\[
P(t) = -S(t) \Phi(-d_1) + KB(t,T) \Phi(-d_2),
\]

where \( K \) is the strike price, \( T \) is the maturity date, \( \Phi(\cdot) \) denotes the cumulative standard normal distribution,

\[
d_1 = \frac{\ln \left( \frac{F_s(t,T)}{K} \right) + \frac{1}{2} \int_t^T \sigma^2_s(u,T) du}{\int_t^T \sigma^2_s(u,T) du} = d_2 + \sqrt{\int_t^T \sigma^2_s(u,T) du},
\]

and \( \sigma^2_{F_s}(t,T) = \sigma^2_s(t) - 2\sigma_s(t)b(t,T)\rho_{SB} + b^2(t,T) \).

First of all, if the risk-free interest rate is a constant, i.e., \( B(t,T) = \exp(-r(T-t)) \), and \( \sigma_s(t) \) is also a constant, then the formula will reduce to the
traditional Black-Scholes pricing formula. By means of the forward price of the
underlying stock, two uncertainty sources, including the underlying stock price and the
default-free ZCB price, can be simplified as one uncertainty source, namely, the
forward price of the underlying stock. Consequently, the pricing formulas only involve
one-dimensional standard normal distributions which show the probabilities that the
options will be in-the-money.

Next, we use Assumptions 1-5 to derive the closed-form pricing formulas for the
vulnerable calls and puts which only suffer from the default risk of the option issuer at
the maturity of the option. Let \( \hat{C}(t) \) and \( \hat{P}(t) \) denote the values of these vulnerable
options and their corresponding final payoffs are given as follows:

\[
\hat{C}(T) = (S(T) - K)^+ 1_{\{V(T) > D\}} + \frac{(1-\alpha)V(T)}{D} (S(T) - K)^+ 1_{\{V(T) \leq D\}},
\]

\[
\hat{P}(T) = (K - S(T))^+ 1_{\{V(T) > D\}} + \frac{(1-\alpha)V(T)}{D} (K - S(T))^+ 1_{\{V(T) \leq D\}},
\]

where \( (x)^+ \equiv \max\{x, 0\} \) and \( 1_{\{B\}} \) denotes the indicator function with value 1 if event
B occurs and with value zero otherwise. The first terms of Equations (6) and (7)
represent the payoffs of the options when the option issuer does not default at the
maturity of the option; namely, at the maturity the value of the option issuer’s total asset
\( V(T) \) is greater than the expected default trigger \( D \), and thus the default does not
occur. The second terms of both the equations are the recovery values of the options at
the maturity when the default occurs at the maturity. The recovery value is, therefore,
equal to the nominal claim of the option, \( (S(T) - K)^+ \), multiplied by the recovery rate
\( (1-\alpha)V(T)/D \). The closed-form pricing formulas of the vulnerable call and put in this
case are given in the following proposition.
Proposition 2: The time $t$ values of a vulnerable Black-Scholes call option $\hat{C}(t)$ and put option $\hat{P}(t)$ which only suffer from counterparty default risk at the maturity of the option are respectively given as follows:

$$
\hat{C}(t) = \left\{ S(t)\Phi_2 \left( e_1, e_5, \rho_{F_t F_t} \right) - KB(t,T)\Phi_2 \left( e_2, e_6, \rho_{F_t F_t} \right) \right. \\
\left. + \frac{(1-\alpha)F_y(t,T)}{D} S(t)e^\int_{\sigma_{F_t}^2(u,T)du} \Phi_2 \left( e_3, e_7, -\rho_{F_t F_t} \right) \right\},
$$

$$
\hat{P}(t) = \left\{ -S(t)\Phi_2 \left( -e_1, e_5, -\rho_{F_t F_t} \right) + KB(t,T)\Phi_2 \left( -e_2, e_6, -\rho_{F_t F_t} \right) \right. \\
\left. - \frac{(1-\alpha)V(t)}{D} S(t)e^\int_{\sigma_{F_t}^2(u,T)du} \Phi_2 \left( -e_3, e_7, \rho_{F_t F_t} \right) \right\},
$$

where $\Phi_2(\cdot,\cdot,\cdot)$ denotes the bivariate cumulative standard normal distribution,

$$
e_1 = d_1, \quad e_2 = d_2,
$$

$$
e_3 = \frac{\ln{\frac{F_y(t,T)}{K}} + \frac{1}{2} \int_{t}^{T} \sigma_{F_t}^2(u,T)du + \int_{t}^{T} \sigma_{F_t F_t}^2(u,T)du}{\sqrt{\int_{t}^{T} \sigma_{F_t}^2(u,T)du}} = e_4 + \sqrt{\int_{t}^{T} \sigma_{F_t}^2(u,T)du},
$$

$$
e_5 = \frac{\ln{\frac{F_y(t,T)}{D}} - \frac{1}{2} \int_{t}^{T} \sigma_{F_t}^2(u,T)du + \int_{t}^{T} \sigma_{F_t F_t}^2(u,T)du}{\sqrt{\int_{t}^{T} \sigma_{F_t}^2(u,T)du}} = e_6 + X(t,T) \sqrt{\int_{t}^{T} \sigma_{F_t}^2(u,T)du},
$$

$$
e_7 = \frac{-\ln{\frac{F_y(t,T)}{D}} - \frac{1}{2} \int_{t}^{T} \sigma_{F_t}^2(u,T)du - \int_{t}^{T} \sigma_{F_t F_t}^2(u,T)du}{\sqrt{\int_{t}^{T} \sigma_{F_t}^2(u,T)du}} = e_8 - X(t,T) \sqrt{\int_{t}^{T} \sigma_{F_t}^2(u,T)du},
\[ X(t,T) = \frac{\int_T^t \sigma_{F_sF_v}(u,T)du}{\int_T^t \sigma_{F_v}^2(u,T)du}, \quad \sigma_{F_v}^2(t,T) = \sigma_v^2(t) - 2\sigma_v(t)b(t,T)\rho_{BV} + b^2(t,T), \]

\[ \rho_{F_sF_v}(t,T) = \frac{\int_T^t \sigma_{F_s}(u,T)du}{\sqrt{\int_T^t \sigma_{F_s}^2(u,T)du \int_T^t \sigma_{F_v}^2(u,T)du}}, \]

and

\[ \sigma_{F_sF_v}(t,T) = \sigma_s(t)\sigma_v(t)\rho_{SV} - b(t,T)\sigma_v(t)\rho_{BV} - b(t,T)\sigma_s(t)\rho_{BS} + b^2(t,T). \]

**Proof** See Appendix A.

When the diffusion terms of Equations (1) and (3) are constants and the risk-free interest rate follows Vasicek (1977), then the pricing formulas for the vulnerable call and put can be reduced to those of KI (1999). The intuitions of Equations (8) and (9) are as follows. First, although there are three risk factors in this model, including the underlying stock price, the option issuer’s total asset value and the default-free ZCB price, it can be reduced via forward prices (i.e., by taking the ZCB as a numeraire) to a two-factor case which involves the forward price of the underlying stock and the forward value of the issuer’s total asset. As a result, Equations (8) and (9) just correlate to the above two factors, their return volatilities, covariance and correlation coefficient. In addition, there are four bivariate cumulative standard normal distributions in Equations (8) and (9), respectively. In contrast with the traditional Black-Scholes pricing formula, these four distributions exhibit the joint probabilities of which the option will be in-the-money and whether the option issuer will declare bankruptcy or not at the maturity of the option. The first two bivariate distributions of Equations (8) and (9) show the joint probabilities of which the option will be in-the-money and the option issuer will not default at the maturity. The
last two bivariate distributions of Equations (8) and (9), on the other hand, represent the joint probabilities of which the options will be in-the-money and the option issuer will default at the maturity.

In what follows, we extend the above model via Assumptions 6 and 7 to incorporate the possible default of the option issuer prior to the maturity of the option into the pricing of the vulnerable options.

Let \( C^*(t) \) and \( P^*(t) \) denote the values of the path-dependent vulnerable call and put options at time \( t \) prior to \( T \) on the set \( \{ \tau_D > t \} \), which means the option issuer has not been bankrupt before time \( t \). The final payoffs are respectively given as follows:

\[
C^*(T) = \frac{(1-\alpha) V(\tau_D)}{D} \mathbb{E}^F_t \left[ (S(T)-K)^+ \left| \mathcal{F}_{\tau_D} \right| 1_{\{\tau<\tau_D\} \land \tau \leq T} + (S(T)-K)^+ 1_{\{\tau > \tau_D\} \land \tau > T} \right],
\]

\[
P^*(T) = \frac{(1-\alpha) V(\tau_D)}{D} \mathbb{E}^F_t \left[ (K-S(T))^+ \left| \mathcal{F}_{\tau_D} \right| 1_{\{\tau<\tau_D\} \land \tau \leq T} + (K-S(T))^+ 1_{\{\tau > \tau_D\} \land \tau > T} \right].
\]

We first clarify the relationship between the time of default \( \tau_D \) and \( \min_{s \in [t,T]} F_v(s,T) \), which is the minimum forward price of the option issuer’s total asset over the time period \((t,T)\). The event of \( \{ \tau_D > T \} \) is equivalent to that of \( \{ \min_{s \in [t,T]} F_v(s,T) < D \} \), and both of the events exhibit that the option issuer does not default during the remaining life of the option. The first terms of Equations (10) and (11) are the recovery value of the vulnerable options at the maturity when the option writer triggers the default at time \( \tau_D \) and \( t < \tau_D \leq T \), i.e., the minimum forward price of the option issuer’s total asset is less than or equal to the expected default trigger \( D \) over the time period \((t,T)\). On the other hand, the second terms of Equations (10) and (11) are the values of vulnerable options when the option issuer does not default during the remaining life of the option.

From Assumption 6 and Assumption 7, we have \( D^* \equiv D \cdot B(\tau_D, T) \) and
\[ V(\tau_D) = D \cdot B(\tau_D, T). \] Substituting them into Equations (10) and (11), we can rewrite them as follows:

\[ C^*(T) = (1 - \alpha)E_{F_t}^P \left[ (S(T) - K)^+ \mathbb{1}_{\{T \leq \tau_D \}} \right] + (1 - \alpha)E_{F_t}^P \left[ (S(T) - K)^+ \right]_{\{T > \tau_D \}}, \]

and

\[ P^*(T) = (1 - \alpha)E_{F_t}^P \left[ (K - S(T))^+ \mathbb{1}_{\{T \leq \tau_D \}} \right] + (K - S(T))^+ \mathbb{1}_{\{T > \tau_D \}}. \]

By the forward risk-neutral pricing method and the law of iterated expectation, we have

\[ C^*(t) = B(t, T)E_{F_t}^P \left[ C^*(T) | \mathcal{F}_t \right] = B(t, T)(1 - \alpha)E_{F_t}^P \left[ (S(T) - K)^+ \mathbb{1}_{\{T \leq \tau_D \}} \right] \]

\[ + B(t, T)E_{F_t}^P \left[ (S(T) - K)^+ \right]_{\{T > \tau_D \}} \mathbb{1}_{\{T \leq \tau_D \}} \mathbb{1}_{\{T > \tau_D \}} \]

\[ \triangleq B(t, T)E_{F_t}^P \left[ C^{**}(T) | \mathcal{F}_t \right], \]

and

\[ P^*(t) = B(t, T)E_{F_t}^P \left[ P^*(T) | \mathcal{F}_t \right] = B(t, T)(1 - \alpha)E_{F_t}^P \left[ (K - S(T))^+ \mathbb{1}_{\{T \leq \tau_D \}} \right] \]

\[ + B(t, T)E_{F_t}^P \left[ (K - S(T))^+ \right]_{\{T > \tau_D \}} \mathbb{1}_{\{T \leq \tau_D \}} \mathbb{1}_{\{T > \tau_D \}} \]

\[ \triangleq B(t, T)E_{F_t}^P \left[ P^{**}(T) | \mathcal{F}_t \right]. \]

In views of the above equations, when pricing the vulnerable options at the time \( t < \tau_D \), without loss of generality, we can replace the final payoff \( C^*(T) \) and \( P^*(T) \) with \( C^{**}(T) \) and \( P^{**}(T) \) in calculating \( C^*(t) \) and \( P^*(t) \), respectively, where \( C^{**}(T) \) and \( P^{**}(T) \) can be written as follows.

\[ C^{**}(T) = (S(T) - K)^+ - \alpha (S(T) - K)^+ \mathbb{1}_{\{T \leq \tau_D \}}, \]  \hspace{1cm} (12)

\[ P^{**}(T) = (K - S(T))^+ - \alpha (K - S(T))^+ \mathbb{1}_{\{T \leq \tau_D \}}. \]  \hspace{1cm} (13)

From Equations (12) and (13), the characteristics of \( C^{**}(t) \) and \( P^{**}(t) \) are similar to those of American put options. The value of an American put option can be decomposed into the value of a European put option plus the “early exercise premium” due to the strategic flexibility of the option owner. In our framework, the value of the
vulnerable option can be decomposed into the value of the standard Black-Scholes (non-vulnerable) options minus the “issuer default discount” because of the implicit default flexibility of the option issuer. Moreover, the value of the vulnerable call option can be decomposed as the sum of a non-vulnerable call option and a short position in \( \alpha \) units of down-and-in outside barrier call options according to the definition of the outside barrier options provided by Heynen and Kat (1994). Also, Equation (13) shows that a vulnerable put option can be viewed as a non-vulnerable put option together with a short position in \( \alpha \) units of down-and-in outside barrier put options.

To derive the pricing formulas of the vulnerable call and put options in our framework, we further transform the down-and-in options into the down-and-out options via the in-out parity of the barrier options. A long position in a down-and-out outside barrier call and a down-and-in outside barrier call is equivalent to a non-vulnerable call option, which can be expressed as

\[
\left( S(T) - K \right)^+ \mathbb{1}_{\min_F \{x, T > D\}} + \left( S(T) - K \right)^+ \mathbb{1}_{\min_F \{x, T < D\}} \equiv \left( S(T) - K \right)^+ .
\]

Therefore, a vulnerable call option can also be replicated by a long position in \( (1-\alpha) \) units of the non-vulnerable calls and \( \alpha \) units of the down-and-out outside barrier calls. By the same token, a vulnerable put option can also be replicated by a long position in \( (1-\alpha) \) units of the non-vulnerable puts and \( \alpha \) units of the down-and-out outside barrier puts. Accordingly, the pricing formulas of the vulnerable call and put options are presented below.

**Proposition 3:** The time \( t \) values of the vulnerable Black-Scholes call option \( C^*(t) \) and put option \( P^*(t) \), on the set \( \{\tau_D > t\} \), which means the option issuer has not been bankrupt before time \( t \), are respectively given as follows:
\[ C'(t) = (1 - \alpha) \left( S(t)\Phi(h_1) - KB(t, T)\Phi(h_2) \right) + \alpha \left\{ S(t)\Phi_2 \left( h_1, h_5, \rho_{F_i}, \rho_{F_i} \right) - KB(t, T)\Phi_2 \left( h_2, h_6, \rho_{F_i}, \rho_{F_i} \right) \right\} \]

\[ - S(t) \frac{F_v(t, T)}{D} \left( \frac{D}{F_v(t, T)} \right)^{2X(t,T)} \Phi_2 \left( h_3, h_5, \rho_{F_i}, \rho_{F_i} \right) \]

\[ + KB(t, T) \frac{F_v(t, T)}{D} \Phi_2 \left( h_4, h_5, \rho_{F_i}, \rho_{F_i} \right) \]

\[ + S(t) \frac{F_v(t, T)}{D} \Phi_2 \left( -h_1, h_7, \rho_{F_i}, \rho_{F_i} \right) \]

\[ - KB(t, T) \frac{F_v(t, T)}{D} \Phi_2 \left( -h_2, h_7, -\rho_{F_i}, \rho_{F_i} \right) \}

\[ (14) \]

\[ P'(t) = (1 - \alpha) \left( -S(t)\Phi(-h_1) + KB(t, T)\Phi(-h_2) \right) + \alpha \left\{ -S(t)\Phi_2 \left( -h_1, h_5, -\rho_{F_i}, \rho_{F_i} \right) + KB(t, T)\Phi_2 \left( -h_2, h_6, -\rho_{F_i}, \rho_{F_i} \right) \right\} \]

\[ + S(t) \frac{F_v(t, T)}{D} \Phi_2 \left( -h_3, h_5, -\rho_{F_i}, \rho_{F_i} \right) \]

\[ - KB(t, T) \frac{F_v(t, T)}{D} \Phi_2 \left( -h_4, h_5, -\rho_{F_i}, \rho_{F_i} \right) \}

\[ (15) \]

\[ \text{where} \quad h_1 = d_1, \quad h_2 = d_2, \]

\[ h_3 = \frac{\ln \left( \frac{F_s(t, T)}{K} \right) + \frac{1}{2} \int_{t}^{T} \sigma_{F_i}^2(u, T)du + \ln \left( \frac{D}{F_v(t, T)} \right)^{2X(t,T)} \sqrt{\int_{t}^{T} \sigma_{F_i}^2(u, T)du}}{\int_{t}^{T} \sigma_{F_i}^2(u, T)du} = h_3 + \sqrt{\int_{t}^{T} \sigma_{F_i}^2(u, T)du}, \]

\[ h_5 = \frac{\ln \left( \frac{F_v(t, T)}{D} \right) - \frac{1}{2} \int_{t}^{T} \sigma_{F_i}^2(u, T)du + \int_{t}^{T} \sigma_{F_i,F_i}^2(u, T)du \sqrt{\int_{t}^{T} \sigma_{F_i}^2(u, T)du}}{\int_{t}^{T} \sigma_{F_i}^2(u, T)du} = h_5 + X(t,T) \sqrt{\int_{t}^{T} \sigma_{F_i}^2(u, T)du}, \]

\[ h_7 = \frac{-\ln \left( \frac{F_v(t, T)}{D} \right) - \frac{1}{2} \int_{t}^{T} \sigma_{F_i}^2(u, T)du + \int_{t}^{T} \sigma_{F_i,F_i}^2(u, T)du \sqrt{\int_{t}^{T} \sigma_{F_i}^2(u, T)du}}{\int_{t}^{T} \sigma_{F_i}^2(u, T)du} = h_7 + X(t,T) \sqrt{\int_{t}^{T} \sigma_{F_i}^2(u, T)du}, \]

\[ \text{and other notations are defined as before.} \]

\[ \text{Proof} \quad \text{See Appendix B.} \]
Equations (14) and (15) will reduce to the Black-Scholes pricing formula under stochastic interest rates when \( D = 0 \), i.e., there is no option issuer’s default risk under the condition \( V(0) > 0 \). If \( \rho_{SV} = 0 \), then our pricing formula becomes very similar that of to Hull and White (1995). At the maturity of the option, Equations (14) and (15), as expected, are respectively equivalent to Equations (8) and (9), which only consider the default at the maturity. In addition, the constant value of the option issuer’s total liability will be extended to follow a stochastic process\(^1\) in Section 5.

The intuitions of Equations (14) and (15) are as follows. Although Equations (14) and (15) consider the possible default of the option issuer during the remaining life of the option, the desired formulas only consist of one-dimensional and two-dimensional cumulative standard normal distributions. These formulas, therefore, are no more complex than the pricing formulas which allow for the default only at the maturity. In Equations (14) and (15), the one-dimensional distributions can be viewed as those of the traditional Black-Scholes pricing formulas under stochastic interest rates, i.e., they are the probabilities that the option will be in-the-money. On the other hand, the two-dimensional distributions of Equations (14) and (15) shows the joint probabilities that the option will be in-the-money and whether the option issuer will declare default or not during the remaining life of the option, which can be derived by applying some techniques of probability theory, as shown in Appendix B.

After deriving the desired formulas, we launch into the hedging strategies of the vulnerable options. In view of Equation (12), we can hold a vulnerable call option together with a long position in \( \alpha \) units of down-and-in outside barrier call options to replicate a non-vulnerable call option. For those hedged fund managers or institutional

\(^1\) For example, Ammann (2001) assumes it follows a geometric Brownian motion.
investors who hold a great position on calls and puts issued by a single option writer, they can request other financial institutions to issue the desired tailor-made down-and-in outside barrier call and put options. After hedging the default risk, the option holders can directly apply the standard hedging strategies of the Black-Scholes options. Notice that the tailor-made down-and-in outside barrier option can be viewed as some kind of credit-linked options, whose holders obtain an option on the underlying stock at the maturity when the pre-specified default event occurs, or else get nothing back. In our model, the default event can be specified as that the forward value of the option issuer’s total asset has ever gone down and touched \( D \) during the remaining life of the option. Moreover, the value of a down-and-in outside barrier option is, in fact, equal to the price difference between the non-vulnerable option and the vulnerable call option.

4. Numerical Examples

In this section, we provide the numerical analyses about the values of the non-vulnerable calls and the vulnerable calls of KI (1999) and our model. Then we also point out some properties of the credit-linked call option employed to hedge the default risk of the option issuer.

Similar to the Black-Scholes option pricing formula under stochastic interest rates, the value of the vulnerable call option is dependent on the underlying stock price, \( S(t) \), the strike price, \( K \), the default-free ZCB price, \( B(t,T) \), the time-to-maturity, \( T-t \), the return volatilities of the underlying stock and the default-free ZCB, \( \sigma_s \) and \( \sigma_b \), and the correlation coefficient between the rates of return of the default-free ZCB.

\footnote{We hereby assume that there is no issuer’s default risk on the down-and-in outside barrier option.}
and the underlying stock, $\rho_{bs}$. Moreover, our formula also depends on the option issuer’s total asset value, $V(t)$, the expected market value of the option issuer’s total liability, $D$, the return volatility of the issuer’s total asset, $\sigma_v$, the correlation coefficient between the rates of return of the default-free ZCB and the issuer’s total asset, $\rho_{bv}$, the correlation coefficient between the rates of return of the issuer’s total asset and the underlying stock, $\rho_{vs}$, and the proportional bankruptcy costs, $\alpha$.

As the same as the standard Black-Scholes model, our model shows that higher risk-free interest rate (lower default-free ZCB price) increases the value of vulnerable calls and decreases the value of vulnerable puts, and higher $\sigma_v$ increases both the values of vulnerable calls and puts. Moreover, higher $\sigma_v$ decreases the value of vulnerable call and put options due to the increase of the option issuer’s default risk. Finally, the effect of bankruptcy costs on the value of the vulnerable options, as expected, exhibit that higher $\alpha$ decreases the value of the vulnerable calls and puts.

Next, we focus on the vulnerable call option. Figure 1 illustrates the intrinsic values of the call, $C_{in}$, the Black-Scholes call prices under stochastic interest rates (Equation (4)), $C_{BS}$, the vulnerable call prices of KI (1999) (Equation (8)), $C_{KI}$, and the vulnerable call prices of our model (Equation (14)), $C_{LH}$, as a function of the prices of the underlying stock. In common with the Black-Scholes call option formula, $C_{KI}$, and $C_{LH}$ are increasing as the price of the underlying stock increases, and are shaped in the same way. As expected, $C_{BS}$ (non-vulnerable) is greater than $C_{KI}$ which considers the default risk only at the maturity, and moreover, $C_{KI}$ is greater than $C_{LH}$ which allows for the possible default during the remaining life of the option. All the parameters, taken from KI (1999), are provided in the footnote of Figure 1, and are served as the base case in Table 1. As noted in KI (1999), the deep-in-the-money vulnerable options may
be worth less than the intrinsic value of the options since both the time value and the intrinsic value of the vulnerable options are suffered from the default risk of the option writer.

Table 1 represents a number of numerical examples for $C_{BS}$, $C_{KI}$ and $C_{LH}$, which are provided to show the effects on the option values under various correlation coefficients among the rates of return of the underlying asset, the default-free ZCB, and the option issuer’s total asset. Panel 1, the base case of which the parameters follow KI (1999), illustrates that the value of the vulnerable call of KI (1999) is underestimated by about 2.1% (relative to our model) due to the neglect of the possibility that the option issuer defaults prior to the maturity. Compared to Panel 1, Panel 2 represents that the effect of the interest rate risk on the two vulnerable call values is the same as that on the non-vulnerable call value, that is, the presence of interest rate risk will raise all the call prices.

Panel 3 exhibits the effects of $\rho_{VS}$ on the values of the vulnerable calls. Intuitively, the values of the non-vulnerable call are unchanged due to the independence of the total asset value of the option issuer. Moreover, higher value of $\rho_{VS}$ induces greater values of both the vulnerable calls. This is because the higher underlying stock price and option issuer’s total asset value will both increase the value of the vulnerable call. Consequently, when $\rho_{VS}$ rises, the vulnerable calls will suffer from smaller default risk. In addition, one may be surprised that when $\rho_{VS} = -0.5$ in Panel 3 of Table 1, the vulnerable call value of our model is greater than that of KI (1999). To be more concrete to explain, we highlight the most important distinction between KI (1999) and our model, and show that one can not directly infer that our model suffers from more default risk of the issuer than KI (1999). Our model takes the possibility of early default into consideration, while the recovery rate is a constant at the maturity of the
option. On the other hand, the recovery rate of KI (1999) is positively related to \( V(T)/D \), the inverse of the debt ratio of the option issuer at the maturity,\(^3\) whereas default may occur only at the maturity. When the debt ratio of the option issuer is rising, the recovery rate of KI (1999) is decreasing, showing that the option holder suffers from increasing default risk. However, this effect will not appear in our framework because our recovery rate is a constant.

We provide Table 2 and Figure 2 to further explore the above surprising result. Table 2 exhibits the price differences in percentage of the vulnerable calls between KI (1999) and our model. Under higher debt ratio of the option issuer, longer time to maturity of the option, and highly negative correlation between the rates of return of \( V \) and \( S \), the surprising result appears, i.e., \( C_{LH} \) is greater than \( C_{KI} \). For example, when \( D/V(t) = 0.9 \), \( \rho_{VS} = -0.8 \) and \( T-t = 3 \), \( C_{KI} \) is underestimated by about 8.8% relative to our model. We can provide a more clear-cut illustration in Figure 2, which shows the values of the two vulnerable calls under various debt ratios and values of \( \rho_{VS} \) when the time to maturity of the option equals to three years. In Figure 2, we can find that either greater debt ratio or highly negative correlation between the returns of \( V \) and \( S \) leads to the reduction of both the two vulnerable call values. When the debt ratio is higher and the value of \( \rho_{VS} \) is smaller, the decreasing speed of \( C_{KI} \) is greater than \( C_{LH} \), and \( C_{KI} \) falls below \( C_{LH} \) as the value of \( \rho_{VS} \) is lower than -0.25. The main reason is that the recovery rate of KI (1999) is negatively related to the debt ratio of the option issuer. Therefore, when the debt ratio increases from 0.5 to 0.9, \( C_{KI} \) falls sharply and becomes lower than \( C_{LH} \). In summary, this surprising phenomenon appears especially when the debt ratio is high, the time to maturity is long and \( \rho_{SV} \) is highly

\(^3\) Therefore, the option issuer’s asset of KI (1999) is allowed to recover from default by maturity.
negatively correlated.

Panel 4 in Table 1 represents the effect of $\rho_{BV}$ on call values. Certainly, the non-vulnerable call value is also unchanged because its value is independent of the value of the option issuer’s total asset. Higher value of $\rho_{BV}$, other things being equal, results in a slightly negative impact on $C_{KI}$ and a positive impact on $C_{LH}$, which are both small. Figure 3 further shows the values of the two vulnerable calls under various debt ratios of the option issuer and values of $\rho_{BV}$ when the time to maturity of the option equals to three years. In view of Figure 3, the effect of $\rho_{BV}$ on the price differences of the two vulnerable calls is small, but the price differences between the two vulnerable calls and the non-vulnerable call are still significant, especially when the debt ratio is high. Moreover, $C_{LH}$ is always less than $C_{KI}$ under various values of $\rho_{BV}$ even if the debt ratio of the option issuer is high.

Panel 5 in Table 1 exhibits the effect of $\rho_{BS}$ on the values of the non-vulnerable call and the two vulnerable calls. It is clear that the higher the value of $\rho_{BS}$, the lower the values of the non-vulnerable call and the two vulnerable calls. In Figure 4, the impacts of $\rho_{BS}$ on the prices of all the call options are in the same manner, i.e., the slopes of all the call options are similar. Figure 4 also shows that higher debt ratio (higher default risk of the issuer) will result in greater reductions of the two vulnerable call values. Again, notice that $C_{LH}$ is always less than $C_{KI}$ under various values of $\rho_{BS}$ even though the debt ratio of the option issuer is high.

Panels 6 and 7 in Table 1 represent the effects of various combinations of $\rho_{VS}$, $\rho_{BV}$ and $\rho_{BS}$. The impacts of these correlations on the value of the non-vulnerable call and the two vulnerable calls are the same as the above analyses. As expected, $\rho_{VS}$
seems to be the most important factor in valuing vulnerable options while the other two correlations also have significant impact on the values of vulnerable options particularly when the debt ratio of the option writer is high and the option is out of money.

Table 3 illustrates the values of credit-linked call options under various correlation coefficients and volatilities of the rates of return. The value of a credit-linked call increases as the price of the underlying stock or the debt ratio of the option writer increases. In addition, higher return volatility of $V$ raises credit-linked call values; higher correlation coefficients between the rates of return of $V$ and $S$ will also make the credit-linked call value greater. Therefore, the credit-linked calls can be used to hedge the default risk of the vulnerable calls. Moreover, the effects of $\rho_{BV}$ and $\rho_{BS}$ on the prices of the credit-linked call option are unstable and small, however. Finally, one may be surprised that in some cases, contrary to the case of Black-Scholes call option, the higher the volatility of the rate of return of the underlying stock, the lower the value of a credit-linked call option. For example, we can observe in Table 3 when $\rho_{BS} = -0.5$, $\rho_{BV} = 0.5$, $\rho_{VS} = 0.5$ and most important, $\sigma_{V} = 0.1$, higher volatility lower credit-linked call values whose values are very small.

5. Extensions

In this section, we generalize our model and allow the value of the option issuer’s total liability to be stochastic. Next, we will apply our pricing framework to value a vulnerable call option on the default-free zero coupon bond.

To allow the value of the option issuer’s total liability to be stochastic, we directly assume that the inverse of the debt ratio of the option writer, $R(t) \equiv V(t)/D(t)$,
follows a geometric Brownian motion, which replaces the setup of $V(t)$ in Assumption 4. Namely, we assume that the dynamics of the inverse of the option writer’s debt ratio under risk-neutrality are given as:

$$dR(t) = R(t)\left(r(t)dt + \sigma_R(t)dW_R(t)\right),$$

where $\sigma_R(t)$ denotes the time-varying volatility of the instantaneous rate of return of $R(t)$, which represents the uncertainty sources of both the issuer’s total asset value and total liability value.\(^4\)

In what follows, we modify the threshold of the option issuer’s total asset value in Assumptions 6 and 7 to the threshold of the inverse of the debt ratio of the issuer, defined by $\gamma$, where $\gamma$ is a positive constant. The time of default is thus changed to

$$\tau_D = \inf\{ t < s \leq T : R(s) \leq 1/\gamma \} = \inf\{ t < s \leq T : \gamma \leq D(s)/V(s) \}.$$  

In other words, the option issuer will default when the issuer’s debt ratio, $D(t)/V(t)$, is greater than or equal to $\gamma$. In addition, the corresponding recovery rate of the vulnerable call at the maturity of the option is altered to $(1-\alpha)/\gamma$.

To price the vulnerable call option in this case, we first provide the final payoff of the vulnerable call with consideration of the stochastic value of the option issuer’s total liability as follows:

$$C^*(T) = \frac{1-\alpha}{\gamma} \left( S(T) - K \right)^+ \mathbb{1}_{\{ \min R(s) \leq \frac{1}{\gamma} \}} + \left( S(T) - K \right)^+ \mathbb{1}_{\{ \min R(s) \geq \frac{1}{\gamma} \}}. \quad (16)$$

Comparing to Equation (10) with the constant value of the issuer’s total liability, we just need the following two substitutions: $(1-\alpha) \rightarrow (1-\alpha)/\gamma$, $D \rightarrow 1/\gamma$ and $F_v(t,T) \equiv V(t)/B(t,T) \rightarrow R(t)$. Hence, the time $t$ value of the vulnerable Black-Scholes call option with stochastic value of the option issuer’s total liability, on

\(^4\) Here we implicitly assume $R(t)$ is a continuously tradable asset.
the set \( \{ \tau_D > t \} \), can be directly obtained by modifying Equation (14) with the above substitutions.

Next, we apply our model to price a vulnerable call option on a default-free zero coupon bond. We first assume that the dynamics of the underlying ZCB price \( B(t, T_1) \) under risk-neutrality, with maturity \( T_1 \) which is larger than \( T \), are given by:
\[
dB(t, T_1) = B(t, T_1) \left( r(t) dt + \sigma_{B_1}(t) dW_{B_1}(t) \right).
\]
Then, all we have to do is to replace the forward price of the underlying stock, \( F_S(t, T) = S(t) / B(t, T) \), with the forward price of the underlying default-free ZCB, \( F_{B_1} = B(t, T_1) / B(t, T) \), in Equation (14). Again, with some modifications, the pricing formula of a vulnerable ZCB call option can be derived.

6. Concluding Remarks

This article derives the closed-form pricing formulas for the Black-Scholes vulnerable options under stochastic interest rates by the forward risk-neutral pricing approach with a time-changed Brownian motion. Our model improves the reduced-form models (Hull and White (1995) and Jarrow and Turnbull (1995)) to consider the correlations among the rates of return of the underlying stock, the option issuer’s total asset and the default-free zero coupon bond. We also extend the previous structural models (Klein (1996) and KI (1999)) to incorporate the possible default of the option issuer during the remaining life of the option, including the maturity of the option. In addition, we provide the specific credit-linked option which can be used to hedge the default risk of the vulnerable options in our model.

In general, the numerical results show that the values of the non-vulnerable options are overestimated by about 10%-15% relative to the vulnerable option values of
our model in most of the cases. The vulnerable call values of KI (1999), which only consider the default risk at the maturity of the option, are also overestimated relative to our model for the most part. It is reversed, however, when the debt ratio of the option issuer is higher, the time to maturity is longer, and the underlying stock price and the issuer’s asset value are highly negatively correlated. In addition, the essential determinants of the vulnerable option values in our model, which are absent in the Black-Scholes model, are the debt ratio of the option issuer, the return volatility of the issuer’s total asset, and the correlation coefficients among the rates of return of the option issuer’s total asset, the underlying stock and the default-free zero coupon bond. In particular, the correlation coefficient between the rates of return of the issuer’s total asset and the underlying stock is the critical factor in the pricing of vulnerable options.

We generalize the model to take the randomness of the option issuer’s total liability into consideration and apply the formula to price a vulnerable call option on a default-free zero coupon bond. Our pricing methodology of the vulnerable options is analogous to those of the risky corporate bonds (such as Longstaff and Schwartz (1995) and Briys and de Varenne (1997)). Therefore, one can directly employ our framework to price the risky derivatives which are allowed to be decomposed into the bond part and the option part, such as bonds with warrant and equity-linked bonds.
Appendix A

Proof of Proposition 2

Under the forward risk-neutral probability measure, i.e., $B(t,T)$ is taken as the numeraire, the forward prices of the underlying stock and the option issuer’s total asset, $F_S(t,T)$ and $F_V(t,T)$, are martingales with respect to the forward risk-neutral filtered probability space $\left( \Omega, \mathcal{F}, \mathbb{P}^T, (\mathcal{F}_t^T)_{t=0}^T \right)$. Their dynamics are respectively given by

$$\frac{dF_S(t,T)}{F_S(t,T)} = \sigma_{F_S}(t,T) dW_{F_S}^T(t), \quad (A1)$$

$$\frac{dF_V(t,T)}{F_V(t,T)} = \sigma_{F_V}(t,T) dW_{F_V}^T(t), \quad (A2)$$

where $W_{F_S}^T(t)$ and $W_{F_V}^T(t)$ are two Wiener processes on $\left( \Omega, \mathcal{F}, \mathbb{P}^T, (\mathcal{F}_t^T)_{t=0}^T \right)$ with the instantaneous correlation coefficient,

$$\rho_{F_S F_V} = \frac{\int_t^T \sigma_{F_S}(u,T) du \int_t^T \sigma_{F_V}(u,T) du}{\sqrt{\int_t^T \sigma_{F_S}^2(u,T) du \int_t^T \sigma_{F_V}^2(u,T) du}}.$$

Similarly, we can employ $S(t)$ as the numeraire and change to the corresponding equivalent probability measure $\mathbb{P}^S$ by Girsanov Theorem. Consequently, under the new filtered probability space $\left( \Omega, \mathcal{F}, \mathbb{P}^S, (\mathcal{F}_t^S)_{t=0}^T \right)$, the dynamics of $F_S(t,T)$ and $F_V(t,T)$ are respectively given as follows:

$$\frac{dF_S(t,T)}{F_S(t,T)} = \sigma_{F_S}(t,T) dt + \sigma_{F_S}(t,T) dW_{F_S}^S(t), \quad (A3)$$

$$\frac{dF_V(t,T)}{F_V(t,T)} = \sigma_{F_V}(t,T) dt + \sigma_{F_V}(t,T) dW_{F_V}^S(t). \quad (A4)$$

According to Equations (A1)-(A4),

$$\mathbb{E}^T\left[ (S(T) - K) 1_{S(T) > K, V(T) > D} \right] | \mathcal{F}_t.$$
\[ = F_S(t,T)P^S(\mathcal{F}) - K \Phi_2(e_2, \rho_{F_{Fv}, F_{V}}) \cdot (A5) \]

By the same way,

\[ \frac{dF_S(t,T)}{F_S(t,T)} = \sigma_{F_{F_{Fv}}, F_{V}}(t,T)dt + \sigma_{F_{V}, F_{V}}(t,T)dW_{F_{v}}^{\rho_V}(t), \quad (A6) \]

\[ \frac{dF_{F_{V}}(t,T)}{F_{F_{V}}(t,T)} = \left( \sigma_{F_{F_{Fv}}, F_{V}}(t,T) + \sigma_{F_{V}, F_{V}}^2(t,T) \right)dt + \sigma_{F_{V}, F_{V}}(t,T)dW_{F_{v}}^{\rho_V}(t), \quad \text{and} \quad (A7) \]

\[ \frac{dF_{F_{Fv}}(t,T)}{F_{F_{Fv}}(t,T)} = \left( \sigma_{F_{F_{Fv}}, F_{F_{Fv}}}(t,T) \right)dt + \sigma_{F_{F_{Fv}}, F_{F_{Fv}}}(t,T)dW_{F_{Fv}}^{\rho_{Fv}}(t). \quad (A8) \]

\[ \frac{dF_{F_{Fv}}(t,T)}{F_{F_{Fv}}(t,T)} = \left( \sigma_{F_{F_{Fv}}, F_{F_{v}}}(t,T) + \sigma_{F_{F_{Fv}}, F_{F_{Fv}}}(t,T) \right)dt + \sigma_{F_{F_{v}}, F_{F_{v}}}(t,T)dW_{F_{Fv}}^{\rho_{Fv}}(t). \quad (A9) \]

According to Equations (A6)-(A9),

\[
\mathbb{E}^P \left[ V(T) \left( S(T) - K \right) 1 \bigg| _{S(T) \geq K, V(T) \leq 0} \left| \mathcal{F}_t \right. \right] \\
= F_S(t,T)F_{V}(t,T)e^{\int_{\sigma_{F_{V}, F_{Fv}}(a, T)\text{d}a}^{\tau} \Phi_2(e_3, e_6, -\rho_{F_{Fv}, F_{V}}) - K \int_{\sigma_{F_{Fv}, F_{Fv}}(a, T)\text{d}a}^{\tau} \Phi_2(e_3, e_6, -\rho_{F_{Fv}, F_{V}}). \quad (A10) \]

From Equations (A5) and (A10), we can apply the forward risk-neutral pricing method to derive the pricing formula of the vulnerable call option \( \hat{C}(t) \). Finally, by means of the put-call parity, we can obtain the valuation formula of the vulnerable put option \( \hat{P}(t) \).
Appendix B

Proof of Proposition 3

To prove Proposition 3, we rewrite the final payoff of the vulnerable call option as

\[
C^*(T) = (1 - \alpha)\left(S(T) - K\right)^+ - \alpha \left(S(T) - K\right)^+ \min_{\tau \leq T} \mathbb{1}_{V(t) > D}, \tag{B1}
\]

We first summarize some basic results concerning the functional of a standard Brownian motion with a constant drift term. The first well-known result, which is commonly referred to as an application of the reflection principle of a standard Brownian motion (see Harrison (1985), P.15, or Musiela and Rutkowski (1997), P.470), is provided in the following lemma.

**Lemma 1:** Let \( Y_1(t) = \nu_1 t + \sigma_1 W_1(t) \) be a standard Brownian motion with a constant drift, where \( \nu_1 \in \mathbb{R} \) and \( \sigma_1 > 0 \), and define \( \tau = \inf(t \geq 0 : Y_1(t) < 0) \) and \( Z(T) = \min\{Y_1(t) : 0 \leq t \leq T\} \). Then for two constants \( y > z \), we have

\[
\Pr\{Z(T) > z, Y_1(T) \geq y\} = \Phi\left(-\frac{y + \nu_1 T}{\sigma_1 \sqrt{T}}\right) - e^{\frac{2\nu_1 z}{\sigma_1^2}} \Phi\left(-\frac{y + 2zT + \nu_1 T}{\sigma_1 \sqrt{T}}\right). \tag{B2}
\]

Next, for another correlated Brownian motion with a constant drift, defined by \( Y_2(t) = \nu_2 t + \sigma_2 W_2(t) \), where \( \rho \) denotes the correlation coefficient between these two standard Brownian motions, we can make the following orthogonal transformation:

\[
W_1 \equiv W \quad \text{and} \quad W_2 \equiv \rho W + \sqrt{1 - \rho^2} \tilde{W},
\]

where \( W \) and \( \tilde{W} \) denote the two independent standard Brownian motions on the same filtered probability space. Since \( Y_2(T) \) is independent of the admixture of \( Z(T) \) and \( Y_1(T) \), given \( Y_1(T) = y \), we have
\[
\Pr[Z(T) > z, Y_z(T) \geq x] = \int_z^\infty \Pr[Y_z(T) \geq x \mid Y_t(T) = y] \Pr[Z(T) > z, Y_t(T) \in dy].
\]

Differentiating (B2) with respect to \( y \) yields \( \Pr[Z(T) > z, Y_t(T) \in dy] \) and

\[
\Pr[Y_z(T) \geq x \mid Y_t(T) = y] = \Phi \left[ \frac{-x + v_z T + \frac{\sigma_z}{\sigma_t} \rho(y-v_t T)}{\sqrt{(1-\rho^2)T}} \right].
\]

Again, by Girsanov Theorem, we have the following result:

\[
E\left[ e^{\alpha_2(T)\int_{(Z(T) > z, Y_z(T) \geq 1)}} \right] = e^{\frac{1}{2} \int_{0}^{T} \sigma_t^2 v_t^2 + \alpha_2 T} \Phi_2 \left( \frac{-z + (v_t + \rho \alpha_1 \sigma_1) T}{\sigma_t \sqrt{T}}, \frac{-x + (v_t + \alpha_2 \sigma_1^2) T}{\sigma_t \sqrt{T}}, \rho \right)
\]

\[
- e^{\frac{1}{2} \int_{0}^{T} \sigma_t^2 v_t^2 + \alpha_2 T} \frac{2(v_t + \rho \alpha_1 \sigma_1) z}{\sigma_t^3} \Phi_2 \left( \frac{-x + (v_t + \alpha_2 \sigma_1^2 + 2 \rho \sigma_z \sigma_1 z) T}{\sigma_t \sqrt{T}}, \frac{-z + (v_t + \rho \alpha_1 \sigma_1) T}{\sigma_t \sqrt{T}}, \rho \right).
\]

(B3)

Since the coefficients of SDEs of our model are all time-varying, we employ the time-changed Brownian technique to transform them into time-independence. For example, define \( Y(t) = -\frac{1}{2} \int_{0}^{t} \sigma_t^2 (u,t) du + \int_{0}^{t} \sigma_{\hat{v}_t}(u,t) dW_{\hat{v}_t}(u) \), and we can see that if \( \sigma_{\hat{v}_t}(\cdot, \cdot) \) satisfies the usual conditions (such as \( \int_{0}^{t} \sigma_{\hat{v}_t}(u,t) du < \infty \)), then

\[
\int_{0}^{t} \sigma_{\hat{v}_t}(u,t) dW_{\hat{v}_t}(u)
\]

is a martingale, \( \mathbb{P}^T \)-a.s. We can define another stopping time as \( T = \inf \{ t > 0 : A(t) \geq t \} \), where \( A(t) \triangleq \int_{0}^{t} \sigma_{\hat{v}_t}(u,t) du \), and apply the time change Brownian motion technique (see Steele (2001), P.203, or Karatzas and Shreve (1991), P.174); consequently, the time-changed process \( \hat{Y}(t) \triangleq Y(A^{-1}(t)) \) can be represented as the Brownian motion with the constant drift term and diffusion term, i.e.,
\[ \hat{Y}(t) = -\frac{1}{2} t + \hat{W}_t(t), \quad \forall t \in [0, A(T)], \] where \( \hat{W}_t \) is a standard Brownian motion with respect to \( \hat{\mathcal{F}}_t \), where \( \hat{\mathcal{F}}_t \triangleq \mathcal{F}_{A^{-1}(t)} \). By the same way, if we define \( \hat{X}(t) = \frac{-1}{2} t + \hat{W}_t(t) \), then we have \( \hat{X}(t) = \frac{-1}{2} t + \hat{W}_t(t) \).

Under the forward probability measure and applying the time change Brownian technique to Equation (B3) with \( \alpha = 1 \), we have

\[
B(0,T)E^{PT} \left[ S(T) 1_{\{S(T) \geq K, \min_{0 \leq s < T} F_s (S, t, D) \}} \right] = S(0)E^{PT} \left[ e^{X(T)} 1_{\{X(T) \geq \ln \frac{K}{F_s(0,T)} \min_{0 \leq s < T} Y(s) > \ln \frac{D}{F_s(0,T)} \}} \right]
\]

\[
= S(0) \Phi_2 \left( h_1, h_5, \rho_{F_s F_v} \right) - S(0) \frac{F_v(0,T)}{D} \left( \frac{D}{F_v(0,T)} \right)^{2X(0,T)} \Phi_2 \left( h_3, h_7, \rho_{F_s F_v} \right),
\]

(B4)

where \( h_1, h_5, h_8, h_7, \) and \( \rho_{F_s F_v} \) are evaluated at \( t = 0 \).

Similarly, when \( \alpha = 0 \), we have

\[
B(0,T)E^{PT} \left[ K 1_{\{S(T) \geq K, \min_{0 \leq s < T} F_s (S, t, D) \}} \right] = KB(0,T)E^{PT} \left[ 1_{\{X(T) \geq \ln \frac{K}{F_s(0,T)} \min_{0 \leq s < T} Y(s) > \ln \frac{D}{F_s(0,T)} \}} \right]
\]

\[
= KB(0,T) \Phi_2 \left( h_2, h_6, \rho_{F_s F_v} \right) - KB(0,T) \frac{F_v(0,T)}{D} \Phi_2 \left( h_4, h_8, \rho_{F_s F_v} \right),
\]

(B5)

where \( h_2, h_4, h_6, h_8 \) and \( \rho_{F_s F_v} \) are evaluated at \( t = 0 \).

Finally, combining Equations (B4) and (B5), we can generalize the pricing formula to the value of a vulnerable call option at time \( t \) by the Markov property of Wiener process. The valuation formula of a vulnerable put option can be derived in a similar way, and thus it is omitted.
References


15. Klein, P., Inglis, M., 2001, Pricing vulnerable European option when the option’s payoff can increase the risk of financial distress, *Journal of Banking and Finance*, 25, 993-1012.


Figure 1. The values of the non-vulnerable and the vulnerable call options under various values of the underlying stock.

This figure compares our model (C_{LH}) to the intrinsic value of the European call (C_{In}), the standard Black-Scholes call option (C_{BS}) and the vulnerable call of KI (1999) (C_{KI}).

The parameters are taken from KI (1999), referred to as the base case. They are as follows: $V(t) = 100$, $D = 90$, $K = 40$, $B(t,T) = 0.8576$, $\alpha = 0.25$, $T - t = 3$, $\sigma_v(t) = 0.2$, $\sigma_S(t) = 0.2$, $b(t,T) = 0.03$, $\rho_{SV} = 0$, $\rho_{BV} = 0$ and $\rho_{BS} = 0$. Notice that in the following numerical analyses, all parameters are the same as the base case, unless otherwise noted.
Figure 2. Values of the Black-Scholes calls and the two vulnerable calls under various debt ratios of the option issuer and values of $\rho_{VS}$.

This figure compares our model ($C_{LH}$) to the standard Black-Scholes call option ($C_{BS}$) and the vulnerable call of KI (1999) ($C_{KI}$).
Figure 3. Values of the Black-Scholes calls and the two vulnerable calls under various debt ratios of the option issuer and values of \( \rho_{BV} \)

This figure compares our model (\( C_{LH} \)) to the standard Black-Scholes call option (\( C_{BS} \)) and the vulnerable call of KI (1999) (\( C_{KI} \)).
Figure 4. Values of the Black-Scholes calls and the two vulnerable calls under various debt ratios of the option issuer and values of $\rho_{BS}$.

This figure compares our model ($C_{\text{LH}}$) to the standard Black-Scholes call option ($C_{\text{BS}}$) and the vulnerable call of KI (1999) ($C_{\text{KI}}$).
Table 1

Values of the European call options under various correlation coefficients

<table>
<thead>
<tr>
<th>Panel</th>
<th>Case</th>
<th>Non-vulnerable call values</th>
<th>Vulnerable call values of KI (1999)</th>
<th>Vulnerable call values of our model</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Base case**</td>
<td>8.49608</td>
<td>7.58423(10.7%)</td>
<td>7.41140(12.8%)</td>
</tr>
<tr>
<td>2</td>
<td>$\sigma_b = 0$</td>
<td>8.44490</td>
<td>7.52164(10.9%)</td>
<td>7.35928(12.9%)</td>
</tr>
<tr>
<td>3</td>
<td>$\rho_{VS} = 0.5$</td>
<td>8.49608</td>
<td>8.18920(3.6%)</td>
<td>7.83874(7.7%)</td>
</tr>
<tr>
<td></td>
<td>$\rho_{VS} = -0.5$</td>
<td>8.49608</td>
<td>6.74736(20.6%)</td>
<td>6.97952(17.9%)</td>
</tr>
<tr>
<td>4</td>
<td>$\rho_{BV} = 0.5$</td>
<td>8.49608</td>
<td>7.58304(10.7%)</td>
<td>7.42862(12.6%)</td>
</tr>
<tr>
<td></td>
<td>$\rho_{BV} = -0.5$</td>
<td>8.49608</td>
<td>7.59106(10.6%)</td>
<td>7.39762(12.9%)</td>
</tr>
<tr>
<td>5</td>
<td>$\rho_{BS} = 0.5$</td>
<td>8.14537</td>
<td>7.16076(12.1%)</td>
<td>7.04100(13.6%)</td>
</tr>
<tr>
<td></td>
<td>$\rho_{BS} = -0.5$</td>
<td>8.82582</td>
<td>7.98216(9.6%)</td>
<td>7.76236(12.1%)</td>
</tr>
<tr>
<td>6</td>
<td>$\rho_{BS} = 0.5; \rho_{BV} = 0.5$</td>
<td>8.14537</td>
<td>7.14847(12.2%)</td>
<td>7.04735(13.5%)</td>
</tr>
<tr>
<td></td>
<td>$\rho_{BS} = -0.5; \rho_{BV} = -0.5$</td>
<td>8.82582</td>
<td>7.99030(9.5%)</td>
<td>7.78955(11.7%)</td>
</tr>
<tr>
<td>7</td>
<td>$\rho_{VS} = \rho_{BV} = \rho_{BS} = 0.5$</td>
<td>8.14537</td>
<td>7.81447(4.1%)</td>
<td>7.51748(7.7%)</td>
</tr>
<tr>
<td></td>
<td>$\rho_{VS} = \rho_{BV} = \rho_{BS} = -0.5$</td>
<td>8.82582</td>
<td>7.22227(18.2%)</td>
<td>7.34807(16.7%)</td>
</tr>
</tbody>
</table>

*We assume the call is at-the-money, and the values in parentheses are the price differences in percentage, defined by the non-vulnerable call value minus the vulnerable call value and divided by the non-vulnerable call value.

**The parameters of the base case are given in Figure 1.
<table>
<thead>
<tr>
<th></th>
<th>$\rho_{VS} = -0.8$</th>
<th>$\rho_{VS} = -0.4$</th>
<th>$\rho_{VS} = 0$</th>
<th>$\rho_{VS} = 0.4$</th>
<th>$\rho_{VS} = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{D}{V(t)} = 0.9$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T - t = 0.1$</td>
<td>0.00166</td>
<td>0.00114</td>
<td>0.00067</td>
<td>0.00025</td>
<td>0.00000</td>
</tr>
<tr>
<td>$T - t = 0.5$</td>
<td>1.45261</td>
<td>1.10936</td>
<td>0.79800</td>
<td>0.49610</td>
<td>0.18094</td>
</tr>
<tr>
<td>$T - t = 1$</td>
<td>1.88980</td>
<td>2.05352</td>
<td>1.99259</td>
<td>1.64266</td>
<td>0.94269</td>
</tr>
<tr>
<td>$T - t = 2$</td>
<td>-2.69052</td>
<td>0.77688</td>
<td>2.69928</td>
<td>3.32394</td>
<td>2.69386</td>
</tr>
<tr>
<td>$T - t = 3$</td>
<td>-8.83748</td>
<td>-1.87301</td>
<td>2.33200</td>
<td>4.29955</td>
<td>4.17663</td>
</tr>
<tr>
<td>$\frac{D}{V(t)} = 0.7$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T - t = 0.1$</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>$T - t = 0.5$</td>
<td>0.01817</td>
<td>0.01077</td>
<td>0.00478</td>
<td>0.00101</td>
<td>0.00000</td>
</tr>
<tr>
<td>$T - t = 1$</td>
<td>0.40340</td>
<td>0.25707</td>
<td>0.13911</td>
<td>0.05071</td>
<td>0.000269</td>
</tr>
<tr>
<td>$T - t = 2$</td>
<td>0.93218</td>
<td>0.87837</td>
<td>0.72053</td>
<td>0.43741</td>
<td>0.09916</td>
</tr>
<tr>
<td>$T - t = 3$</td>
<td>-0.57865</td>
<td>0.64513</td>
<td>1.1141</td>
<td>0.95892</td>
<td>0.36176</td>
</tr>
<tr>
<td>$\frac{D}{V(t)} = 0.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T - t = 0.1$</td>
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<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>$T - t = 0.5$</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>$T - t = 1$</td>
<td>0.00247</td>
<td>0.00127</td>
<td>0.00042</td>
<td>0.00004</td>
<td>0.00000</td>
</tr>
<tr>
<td>$T - t = 2$</td>
<td>0.15863</td>
<td>0.08939</td>
<td>0.03922</td>
<td>0.00908</td>
<td>0.00008</td>
</tr>
<tr>
<td>$T - t = 3$</td>
<td>0.41845</td>
<td>0.28917</td>
<td>0.16999</td>
<td>0.06195</td>
<td>0.00227</td>
</tr>
</tbody>
</table>

The price difference in percentage is defined by the vulnerable call value of KI (1999) minus the vulnerable call value of our model and divided by the vulnerable call value of our model. The positive (negative) value displays the overestimation (underestimation) of KI (1999). All the parameters are the same as the base case, unless otherwise noted.
Table 3

Values of the credit-linked call options under various correlation coefficients and volatilities

<table>
<thead>
<tr>
<th></th>
<th>$\rho_{SV}$</th>
<th>$\rho_{BS} = 0.5$</th>
<th>$\rho_{BS} = -0.5$</th>
<th>$\rho_{BV} = 0.5$</th>
<th>$\rho_{BV} = -0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_v = 0.1$, $\sigma_s = 0.1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>5.77342</td>
<td>6.12518</td>
<td>6.06703</td>
<td>5.11079</td>
<td>5.01838</td>
</tr>
<tr>
<td>0.5</td>
<td>14.69170</td>
<td>15.16890</td>
<td>15.00630</td>
<td>14.87620</td>
<td>14.87620</td>
</tr>
<tr>
<td>$\sigma_v = 0.1$, $\sigma_s = 0.3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>5.11079</td>
<td>5.01838</td>
<td>5.01838</td>
<td>4.82065</td>
<td>4.82065</td>
</tr>
<tr>
<td>$\sigma_v = 0.3$, $\sigma_s = 0.3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>14.69170</td>
<td>15.16890</td>
<td>15.00630</td>
<td>14.87620</td>
<td>14.87620</td>
</tr>
<tr>
<td>$\sigma_v = 0.3$, $\sigma_s = 0.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>14.69170</td>
<td>15.16890</td>
<td>15.00630</td>
<td>14.87620</td>
<td>14.87620</td>
</tr>
</tbody>
</table>

The credit-linked call option values are expressed as per unit, and all the parameters are same as the base case, unless otherwise noted.