The Valuation of European Options When Asset Returns Are Autocorrelated

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Abstract

This paper derives the closed-form formula for a European option on an asset with returns following a continuous-time type of first-order moving average process, which is named as an MA(1)-type option. The pricing formula of these options is similar to that of Black and Scholes except for the total volatility input. Specifically, the total volatility input of MA(1)-type options is the conditional standard deviation of continuous-compounded returns over the option’s remaining life, whereas the total volatility input of Black and Scholes is indeed the diffusion coefficient of a geometric Brownian motion times the square root of an option’s time to maturity. Based on the result of numerical analyses, the impact of autocorrelation induced by the MA(1)-type process is significant to option values even when the autocorrelation between asset returns is weak.

Key words: European Option Pricing; Autocorrelated Returns; Martingale Asset Pricing

JEL classification: G13
1 Introduction

As pointed out in Lo and Wang (1995), there is now a substantial body of evidence that documents the predictability of financial asset returns. In addition to the mean-reverting model, the moving average process is one popular model to describe predictable financial asset returns. Precisely, to capture the autocorrelation of financial asset returns, many documents extract the autocorrelation from the asset returns’ first moment by the form of a first-order moving average process (MA(1) process), including Hamao, Masulis, and Ng (1990), Bollerslev (1987), and French, Schwert, and Stambaugh (1987), to name a few.

It is also well known that the value of an option may depend on the underlying asset’s log-price dynamics. The famous Black-Scholes model assumes that the stock price process is a geometric Brownian motion, which implies stock returns are independent. In distinguishing between the risk-neutral and true distributions of an option’s underlying asset return process, Grundy (1991) shows that the Black-Scholes formula still holds even though the underlying asset returns follow an Ornstein-Uhlenbeck process. Along this line of research, Lo and Wang (1995) claim that the unconditional variance of returns is usually fixed for any given set of data irrespective of predictability. Accordingly, when one implements pricing formulae of options on assets with predictable returns, the values of the pricing formulae’s parameters should be adjusted to fit the unconditional moments of returns. Based on the preceding assertion and the result of Grundy (1991), Lo and Wang (1995) further price options on an asset with the trending Ornstein-Uhlenbeck process (trending O-U process) by
using the Black-Scholes formula with an adjustment for predictability. Apparently, before implementing an option pricing formula by the approach of Lo and Wang (1995), the pricing formula for the used model should be known in advance. As shown in Lo and Wang (1995), an important result of the arbitrage-free methods for pricing derivatives is: as long as the underlying asset’s log-price dynamics are described by an Itô diffusion process with a constant diffusion coefficient, the Black-Scholes formula yields the correct option price regardless of the specification and arguments of the drift. However, there exist no research studies concerning the extent to whether the Black-Scholes formula still holds when the underlying asset returns are described by an MA(1) process. Thus, the main objective in the paper herein is to fill the gap by introducing a continuous-time MA(1)-type process, which is consistent with the findings in empirical studies, and to price European options on an asset with the process by using the martingale pricing method.

The underlying asset’s log-price dynamics in the paper are similar to a special case of the discrete-time model used in Jokivuolle (1998). Specifically, Jokivuolle (1998) values a European option on observed stock index returns which are specified as an infinity-order moving average process and are assumed to be different from true index returns. However, unlike Jokivuolle (1998), this paper assumes that the process of asset returns is a continuous-time type and the observed and true returns are identical, which is a common assumption in the martingale pricing methods.

One contribution of the paper is to price European options on an asset with continuous-time MA(1)-type dynamics (MA(1)-type options) by the martingale method. As a result,
it is found that the pricing formula of MA(1)-type options is not identical to that of Black and Scholes. Accordingly, numerical analyses are conducted herein to gauge the impact of autocorrelation induced by the MA(1)-type process on option values.

The remaining parts of this paper are organized as follows. Section 2 shows the setting of a continuous-time process for autocorrelated asset returns considered in this paper. Section 3 illustrates the pricing formula and the hedge for the MA(1)-type options. Section 4 and section 5 provide the results of numerical analyses and a conclusion, respectively.

2 The Setting: A Continuous-time Process of Autocorrelated Asset Returns

Without loss of generality, this paper considers the underlying asset to be a stock and denotes the underlying stock price including dividends as $S$. The current time is $t_0$, the expiration date of the options considered here is $T$, and the time to maturity is $\tau$, where $\tau = T - t_0$ and $\tau > 0$. As the stock returns of a first-order moving average process are a common finding in empirical research studies, this paper introduces a continuous-time MA(1) process (MA(1)-type process) and assumes the dynamics of the stock price as follows:

$$
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + \beta \sigma dW_{t-h},
$$

where $\mu$ is a constant expected appreciation rate of the stock price, $\sigma > 0$ is a constant volatility coefficient, $dt > 0$ is an infinitesimal time interval, and $h > 0$ is a fixed, but arbitrary, small constant. The coefficient $\beta$ represents the impact of the past shock, which is assumed to satisfy $|\beta| < 1$. As the condition of $|\beta| < 1$ is a standard assumption for a
discrete-time invertible MA(1) representation, there is no loss of generality when imposing
the assumption here. In addition, $W_t$ is a one-dimensional standard Brownian motion defined
on a naturally filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0,T]})$ and $dW_{t-i}, i = 0, h,$ are the
increments of the standard Brownian motion at time $t - i$. In empirical works, $h$ is restricted
by the frequency of historical data.

The dynamics of stock prices in (1) are equivalent to the following Itô integral equation:

$$S_t = S_{t_0} + \int_{t_0}^{t} \mu S_u du + \int_{t_0}^{t} \sigma S_u dW_u + \int_{t_0}^{t} \sigma \beta S_u dW_{u-h}, \quad \forall \ t \in \left[t_0, T\right], \quad (2)$$

where $S_{t_0} \in \mathbb{R}_+$ ($\mathbb{R}_+$ denotes the set of all strictly positive real numbers) is the current stock
price. In addition, the conditional variance of stock returns at time $t$ conditional on the
information set up to time $t_0$ is

$$Var_{t_0}(R_t) = \sigma^2 dt, \quad \forall \ t \in \left[t_0, t_0 + h\right);$$

$$Var_{t_0}(R_t) = (1 + \beta^2)\sigma^2 dt, \quad \forall \ t \in \left[t_0 + h, T\right],$$

and the conditional autocorrelation coefficient is given by

$$Corr_{t_0}(R_t, R_{t+h}) = \frac{\beta}{\sqrt{1 + \beta^2}}, \quad \forall \ t \in \left[t_0, t_0 + h\right);$$

$$Corr_{t_0}(R_t, R_{t+h}) = \frac{\beta}{1 + \beta^2}, \quad \forall \ t \in \left[t_0 + h, T\right],$$

where $R_t$ denotes the stock return at time $t$, i.e., $R_t \equiv dS_t/S_t$. Based on the conditional
variance and autocorrelation coefficient, the main properties of stock returns specified as in
(1) can be clearly observed. Obviously, the stock returns are independent and (1) reduces to
a geometric Brownian motion when $\beta = 0$. For the case of $\beta \neq 0$, the stock returns specified in (1) exhibit non-zero autocorrelation, which can be positive or negative depending upon the sign of $\beta$. Consequently, this process is more flexible than the usual geometric Brownian motion.

Many studies like Bollerslev (1987), French, Schwert, and Stambaugh (1987), and Hamao, Masulis, and Ng (1990) employ the MA(1) process as an empirical model for stock returns $R_t$. Some of them (e.g., Hamao, Masulis, and Ng (1990)) find that the impact of past shocks, such as $\beta$ in Equation (1) herein, is significant. Hence, the continuous-time MA(1)-type process specified in (1) is empirically justified.

The underlying asset’s log-price dynamics in (1) are similar to a special case of the discrete-time model used in Jokivuolle (1998). However, compared to Jokivuolle (1998), where the true index returns are assumed to follow a random walk with drift, but the observed index returns are specified as an infinite-order moving average process, this paper does not distinguish the true stock returns from the observed ones. This setting is based on the common assumption of the martingale pricing method and is more in line with Lo and Wang (1995), where the observed stock returns are identical to the true returns. In addition, it is worth noting that Lo and Wang (1995) assume that stock prices follow a trending O-U process and consequently the autocorrelations in the stock returns are caused by the drift term. In contrast, as shown in Equation (1), the current article proposes a different model for autocorrelated returns, where the autocorrelated behavior comes from the diffusion term.

Two modeling issues concerning Equation (1) have not been discussed until now: (i) Is
the price process specified in (1) conceivably used to represent security price fluctuations? 

(ii) Does the price process specified in (1) admit arbitrage opportunities? For the first issue, Harrison and Pliska (1981, p.222) show that as long as the price process $S$ satisfies the condition that the discounted price process is a martingale under some probability measure $Q$ equivalent to $P$, then the price process can be used to represent security price fluctuations. As for the second issue, it is a well known result that there exist no arbitrage opportunities when $S$ satisfies the preceding condition.\footnote{1} For expositional purposes, the preceding condition will be checked in the next section.

3 Option Pricing When Asset Returns follow an MA(1)-type process

Consider the problem of pricing a European call option on a specified stock as in Equation (1). Since the underlying stock returns are autocorrelated, it is not easy to value the call by tree methods. This paper develops a pricing formula by the martingale methodology.

To price the derivatives, it is more convenient to have a risk-free security. Suppose the short-term interest rate $r$ is constant over the trading interval $[t_0, T]$, and the value of a riskless bond denoted by $B$ is assumed to be continuously compounded at the rate $r$; that is,

$$\frac{dB_t}{B_t} = rdt, \quad \forall \ t \in [t_0, T],$$

or equivalently $B_t = e^{rt}$, with $B_0 = 1$. 

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Based on the risk-neutral pricing theory, the current value of a European call option \( C_{t_0} \) is

\[
C_{t_0} = e^{-r(T-t_0)} E^Q \left\{ \left( S_T - K \right)^+ \mid \mathcal{F}_{t_0} \right\}.
\]

Here, \( Q \) is a martingale measure corresponding to the use of the riskless bond \( B \) as the numeraire, \( E^Q \left\{ \cdot \mid \mathcal{F}_{t_0} \right\} \) denotes the expectation under measure \( Q \) conditional on \( \mathcal{F}_{t_0} \), \( K \) is the strike price of the European call option, and \( (S_T - K)^+ \) is the notation for \( \max(S_T - K, 0) \).

According to the martingale pricing method and (4), pricing the MA(1)-type option is done under the martingale probability measure \( Q \) that makes the discounted stock price \( \bar{S}_t \equiv S_t/B_t \) into a martingale, which can be represented as:

\[
E^Q \left\{ \frac{S_t}{B_t} \mid \mathcal{F}_{t_0} \right\} = \frac{S_{t_0}}{B_{t_0}}, \quad \forall \ t \in [t_0, T].
\]

According to (3), Equation (5) can be rewritten as:

\[
E^Q \left\{ \frac{S_t}{S_{t_0}} \mid \mathcal{F}_{t_0} \right\} = e^{r(t-t_0)}, \quad \forall \ t \in [t_0, T],
\]

which shows that the expected stock returns equal the riskless rate \( r \) under martingale measure \( Q \). Based on the dynamics of the stock price in (1) and the definition of probability measure \( Q \), the transformation from measure \( P \) to \( Q \) is shown in the following Lemma.

Lemma 1. Assume that the underlying asset’s price process \( S \) satisfies Equation (1). Specifically, \( W \) is a \( P \)-Brownian motion. The transformation from \( P \)-Brownian motion
to $Q$-Brownian motion $W^Q$ is then

$$dW^Q_z = dW_z + \sum_{j=0}^{iz} (-\beta)^j (\mu - r) \frac{dz}{\sigma} + (-1)^{iz} \beta^{iz+1} dw_{z-(iz+1)h}, \quad \forall z \in [t_0, T],$$

where $z = t, t - h,$ and $iz$ is the integer part of $(z - t_0)/h$.

**Proof:** See Appendix 1.

Let $\theta_z$ denote $dw_{z-(iz+1)h}$, which in fact are the realized past increments of the Brownian motion. The result of Lemma 1 can be represented as:

$$dW^Q_z = dW_z + H_z dz,$$

where

$$H_z = \sum_{j=0}^{iz} (-\beta)^j (\mu - r) \frac{dz}{\sigma} + (-1)^{iz} \beta^{iz+1} \theta_z, \quad \forall z \in [t_0, T].$$

Specifically, $H_z$ is predictable and can be displayed according to time intervals as follows:

$$H_z = \left(\frac{\mu - r}{\sigma}\right) + \beta \theta_z, \quad \forall z \in [t_0, t_0 + h);$$

$$H_z = \left(\frac{1 - \beta}{\sigma}\right)(\mu - r) + (-1)\beta^2 \theta_z, \quad \forall z \in [t_0 + h, t_0 + 2h);$$

$$H_z = \left(\frac{1 - \beta + \beta^2}{\sigma}\right)(\mu - r) + \beta^3 \theta_z, \quad \forall z \in [t_0 + 2h, t_0 + 3h);$$

$$\vdots$$

$$H_z = \left[\frac{(1 - \beta + \beta^2 - \cdots + (-\beta)^{ir})}{\sigma}\right](\mu - r) + (-1)^{ir} \beta^{ir+1} \theta_z, \quad \forall z \in [t_0 + irh, T].$$

Since the existence of measure $Q$ is assured by Girsanov’s theorem, the prerequisite for Girsanov’s theorem that $H_z$ is a predictable process with $\int_{t_0}^{T} H^2_z ds < \infty$ should be checked.
As the path of a Brownian motion before the current time \( t_0 \) is known, the values of \((-1)^iz^i\beta^i(z^{i+1}h)\) for all \( z \) are bounded under the condition \( |\beta| < 1 \). In addition, since \( h, \mu, r, \sigma, \) and \( \beta \) in (1) are all assumed to be constant, the values of \((z - t_0)/h\) and \[\sum_{j=0}^{i-1}(-\beta)^j(\mu - r)\]/\(\sigma\) are also bounded for all \( z \). Accordingly, the assumption of Girsanov’s theorem, \( \int_{t_0}^T H^2_s ds < \infty \), is satisfied and thus the existence of measure \( Q \) can be assured.

The existence of measure \( Q \) provides enough information to answer the question proposed in Section 1: Does there exist an arbitrary opportunity when stock prices follow (1)? As shown in Klebaner (1998, p.258), the sufficient condition for no arbitrage can be stated as follows: Suppose there exists a probability measure \( Q \), equivalent to \( P \), such that the discounted stock price process \( \tilde{S} \) is a martingale under \( Q \). There are then no arbitrage opportunities. Accordingly, an asset with a process specified in (1) allows no arbitrage opportunities, because the existence of measure \( Q \) is ensured by Girsanov’s theorem.

After recognizing that the log-price dynamics defined in (1) allow no arbitrage opportunities, the MA(1)-type options can be valued by the martingale pricing method, which is done under the martingale probability measure \( Q \). By Lemma 1, the process defined in Equation (1) can be transformed to the dynamics of the stock price under measure \( Q \), denoted as \( S^Q \), as follows:

\[
\frac{dS^Q_t}{S^Q_t} = r dt + \sigma dW^Q_t + 1_A \sigma \beta dW^Q_{t-h}, \quad \forall \ t \in [t_0, T],
\]

where \( 1_A = 1_{\{t_0 + h \leq t \leq T\}} \) is an indicator function that takes a value of 1 for \( t \in [t_0 + h, T] \), and 0 otherwise. Note that the time to maturity \( T - t_0 \) can be distinguished into two cases:
0 < T − t_0 < h and T − t_0 ≥ h. Trivially, for the case of t_0 < T < t_0 + h, the stock price process \( S^Q \) under measure \( Q \) reduces to a geometric Brownian motion. Accordingly, the Black-Scholes formula is still applicable to the MA(1)-type call option with maturity shorter than \( h \).

For the case of \( T − t_0 \geq h \), the price process \( S^Q \) is not a geometric Brownian motion. To value the term \( E \{(S_T − K)^+ | \mathcal{F}_t \} \) in (4), the solution for the stock price at time \( t \) under measure \( Q \), denoted as \( S^Q_t \), should be at hand. To solve \( S^Q_t \), the dynamics of stock prices under \( Q \) displayed in (7) can be viewed as being driven by two Brownian motions, \( W^Q_{1,t−t_0} \) and \( W^Q_{2,t−t_0} \), where \( \{W^Q_{1,t−t_0}, W^Q_{2,t−t_0}\} \equiv \{W^Q_{t−t_0}, 1_A W^Q_{t−(h−t_0)}\} \). The two Brownian motions have the following properties: (i) \( W^Q_{1,t−t_0} \) and \( W^Q_{2,t−t_0} \) are both one-dimensional Brownian motions; (ii) \( W^Q_{1,(t−h)−t_0} = W^Q_{2,t−t_0} \) for \( t \in [t_0 + h, T] \), and \( dW^Q_{1,t−h} = W^Q_{(t−h)−t_0} + dt - W^Q_{(t−h)−t_0} = dW^Q_{2,t} \) for \( t \in [t_0 + h, T] \); (iii) \( dW^Q_{1,t} \) and \( dW^Q_{2,t} \) are uncorrelated, i.e., \( E(dW^Q_{1,t}dW^Q_{2,t}) = 0 \). Apparently, \( dS^Q_t/S^Q_t \) in (7) can also be represented as

\[
\frac{dS^Q_t}{S^Q_t} = r dt + \sigma \left(W^Q_{1} + \beta W^Q_{2}\right)_t.
\]

To solve \( S_t^Q \), the quadratic variation of \( \left(W^Q_{1} + \beta W^Q_{2}\right)_t \), which is denoted as \( \langle W^Q_{1} + \beta W^Q_{2}\rangle_t \), is needed and can be represented as

\[
\langle W^Q_{1} + \beta W^Q_{2}\rangle_t = \int_{t_0}^{t} (1 + 1_{B(s)}\beta)^2 ds,
\]

where \( 1_{B(s)} = 1_{\{t_0 \leq s \leq t−h\}} \). Based on (7) and (8), \( S_t^Q \) can be solved by using Itô’s lemma as follows:

\[
S_t^Q = S_{t_0} e^{\left[\frac{1}{2} \sigma^2 (1+\beta)^2 \right] (t−h)−t_0 + (r−\frac{1}{2} \sigma^2)h + \sigma (1+\beta) W^Q_{(t−h)−t_0} + \sigma W^Q_{t−(t−h)}}, \quad \forall \ t \in [t_0, T].
\]
It is easy to check that $S^Q_t$ in (9) is the solution such that the discounted stock price $\tilde{S}_t$ is a martingale under measure $Q$ for $t \in [t_0, T]$.

To value the MA(1)-type options, it is convenient to find the probability measure $R$ equivalent to $Q$ such that the following equation is satisfied:

$$E^Q\{S_T \cdot 1_{\{S_T > K\}} \mid \mathcal{F}_{t_0}\} = S_{t_0}e^{r(T-t_0)}\text{Prob}^R(S_T > K \mid \mathcal{F}_{t_0}). \tag{10}$$

The existence of such a measure $R$ is assured by Girsanov’s theorem. Define that $W^R_z$ process as

$$dW^R_z = \begin{cases} dW^Q_z - \sigma(1 + \beta)dz, & \forall \ z \in [t_0, T - h]; \\ dW^Q_z - \sigma dz, & \forall \ z \in (T - h, T], \end{cases} \tag{11}$$

where $z = t, t - h$. Term $W^R$ is then a $R$-Brownian motion satisfied Equation (10). To value the term $\text{Prob}^R(S_T > K \mid \mathcal{F}_{t_0})$ in (10), the solution for the stock price at time $t$ under measure $R$, denoted as $S^R_t$, should be at hand. Using (11) and (7), $S^R_t$ can be solved by using Itô’s lemma as follows:

$$\ln S^R_t = \ln S_{t_0} + (r + \frac{1}{2}\sigma^2)h + [r + \frac{1}{2}\sigma^2(1 + \beta)^2][t - (t_0 + h)]$$

$$+ \sigma(1 + \beta)W_{(t-h)-t_0} + \sigma W_{t-(t-h)}. \tag{12}$$

The probabilities that the call is in the money at the maturity date under measures $Q$ and $R$ can accordingly be obtained from the solutions $S^Q_T$ and $S^R_T$, respectively, as

$$\text{Prob}^Q(S_T > K \mid \mathcal{F}_{t_0}) = N\left(\frac{\ln(S_{t_0}) + r(T-t_0) - \frac{1}{2}\sigma^2[(1 + \beta)^2(T-t_0 - h) + h]}{\sigma\sqrt{(1 + \beta)^2(T-h-t_0) + h}}\right) \equiv N(d'_2),$$

and

$$\text{Prob}^R(S_T > K \mid \mathcal{F}_{t_0}) = N\left(\frac{\ln(S_{t_0}) + r(T-t_0) + \frac{1}{2}\sigma^2[(1 + \beta)^2(T-t_0 - h) + h]}{\sigma\sqrt{(1 + \beta)^2(T-h-t_0) + h}}\right) \equiv N(d'_1).$$
Therefore, based on (4), (9), (10), and (12), the current value of the MA(1)-type option, \(C_{t_0}\), is priced by the following:

\[
C_{t_0} = S_{t_0} N(d'_1) - Ke^{-r(T-t_0)} N(d'_2).
\]

This result is summarized in Proposition 1.

**Proposition 1.** Assume that the dynamics of the underlying stock prices are given by (1). The value of a European call option on the preceding underlying stock, which is named as an MA(1)-type option, can be priced by the following:

(i) When the time to maturity satisfies \(T - t_0 < h\), i.e., the call option will mature immediately, the Black-Scholes formula still holds for the MA(1)-type option.

(ii) When the expiration date satisfies \(T \geq t_0 + h\), i.e., the call will not mature immediately, the value of the MA(1)-type option, \(C_{t_0}\), is priced by

\[
C_{t_0} = S_{t_0} N(d'_1) - Ke^{-r(T-t_0)} N(d'_2),
\]

where

\[
d'_1 = \frac{\ln(S_{t_0}) + r(T - t_0) + \frac{1}{2}\sigma^2 [(1 + \beta)^2(T - t_0 - h) + h]}{\sigma \sqrt{(1 + \beta)^2(T - h - t_0) + h}},
\]

\[
d'_2 = \frac{\ln(S_{t_0}) + r(T - t_0) - \frac{1}{2}\sigma^2 [(1 + \beta)^2(T - t_0 - h) + h]}{\sigma \sqrt{(1 + \beta)^2(T - h - t_0) + h}},
\]

and \(N(\cdot)\) is the distribution function of the standard normal distribution.
The formula in Proposition 1 obviously indicates that the MA(1)-type option price will eventually converge to the Black-Scholes price when the call closes to maturity. This result is consistent with the assumption of Roll (1977), Duan (1995), and Heston and Nandi (2000), where they assume that the value of an option with one period to expiration obeys the Black-Scholes formula in discrete-time models. Furthermore, the pricing formula in Proposition 1 does not violate the Black-Scholes formula. Apparently, $C_{t_0}$ converges to the Black-Scholes formula when $h$ closes to zero, and it is fully identical to the Black-Scholes formula as $\beta = 0$.

The pricing formula for the MA(1)-type option is analogous to Black and Scholes except for the total volatility input. Denote the continuously-compounded $\tau$-period returns as $R_\tau$, where $\tau$ represents the option’s time to maturity. When the stock prices satisfy an MA(1)-type process specified as in (1), the conditional variance of $R_\tau$ conditional on the information up to time $t_0$ is

$$Var_{t_0}(R_\tau) = \sigma^2 \left[ (1 + \beta^2)(T - t_0 - h) + h \right].$$

Accordingly, the conditional standard deviation of $R_\tau$ is the square root of $Var_{t_0}(R_\tau)$, which is an important term of $d'_1$ and $d'_2$ as shown in Proposition 1. As the total volatility input in the standard Black-Scholes formula is indeed the diffusion coefficient of a geometric Brownian motion multiplied by the square root of an option’s time to maturity, i.e., $\sigma\sqrt{\tau}$, Proposition 1 shows that the total volatility input for an MA(1)-type option is the conditional standard deviation of $R_\tau$, i.e., $\sqrt{Var_{t_0}(R_\tau)}$, where $Var_{t_0}(R_\tau)$ is displayed in (13). Based on the result of Grundy (1991), i.e., the Black-Scholes formula still holds under the trending O-U
specification, Proposition 1 implies that the *pricing formula* for options on an asset with autocorrelated returns depends on the source of the autocorrelation.

To hedge the European call options under a stock price process specified as in (1), at the current time \( t \in [t_0, t_0 + dt] \), one may consider a portfolio \( V_t \) which consists of \( \Delta_t \) shares of stock and \( \phi_t \) units of riskless bonds, i.e.,

\[
V_t = \Delta_t S_t + \phi_t B_t,
\]

Note that the number of shares in \( V_t \), i.e., \( \Delta_t \), is also called the hedge ratio. Assume that the portfolio replicates the MA(1)-type call option. The changes in the values of the portfolio and the option, denoted as \( dV_t \) and \( dC_t \), respectively, should then be equal. This implies

\[
(\Delta_t \mu S_t + \phi_t r B_t) dt + \Delta_t \sigma S_t dW_t + \Delta_t \sigma \beta S_t dw_{t-h} = \left( \frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S_t} \mu S_t + \frac{1}{2} \frac{\partial^2 C_t}{\partial S_t^2} \sigma^2 S_t^2 \right) dt + \frac{\partial C_t}{\partial S_t} \sigma S_t dW_t + \frac{\partial C_t}{\partial S_t} \sigma \beta S_t dw_{t-h}.
\]

Note that based on the information up to time \( t_0 \), the term \( dw_{t-h} \) is known.

In view of the preceding equation and Proposition 1, it is observed that

\[
\Delta_t = \frac{\partial C_t}{\partial S_t} = N \left( d'_1 \right),
\]

and the number of bonds in the portfolio \( V_t \), i.e., \( \phi_t \), can be decided accordingly. Because portfolio \( V_t \) replicates the value of the MA(1)-type option, it is apparent that hedging the MA(1)-type option can be performed just by holding portfolio \( V_t \) in the opposite position. Thus, hedging the MA(1)-type option is identical to that of the Black-Scholes model in functional form and is easy to operate.
4 Numerical Analyses

To gauge the impact of autocorrelated stock returns on the option’s price, Tables 1 and 2 compare the theoretical values of options under an MA(1)-type process to the Black-Scholes prices for various times to maturity $T - t_0$, strike prices $K$, and autocorrelated parameters $\beta$ for a hypothetical $40$ stock. The theoretical values of MA(1)-type options are calculated by the result of Proposition 1, and the Black-Scholes values are based on the Black-Scholes formula. It is worth noting that the design of numerical analyses in this paper is similar to that of Jokivuolle (1998), where both papers compare the MA option values to the Black-Scholes prices based on the same value of $\sigma$. For both Tables 1 and 2, the coefficient for the current increment of a Brownian motion, i.e., $\sigma$, is set to be 1.75 percent per day. Accordingly, unlike Lo and Wang (1995), the variance of daily returns under an MA(1)-type process is different from the geometric Brownian motion’s counterpart.

Panel A of Tables 1 and 2 show that even extreme autocorrelated parameters do not affect short-maturity in-the-money call option prices very much. To illustrate, as shown in Table 1, the value of $\beta$ has no impact on the $30$ 7-day call even when $\beta$ is equal to -0.75. A similar pattern is found as $\beta$ equals 0.75, where the price under the MA(1)-type process is identical to the standard Black-Scholes price of $10.014$. For all of the short-maturity call options, Tables 1 and 2 also exhibit that the MA(1)-type option price converges to the Black-Scholes price in absolute terms, which is consistent with the result of Proposition 1. However, the impact of autocorrelated parameter $\beta$ grows with the length of time to maturity. As shown in
Table 1, when $\beta$ is equal to -0.75, the absolute difference between the Black-Scholes price and the MA(1)-type price for the $30$ 364-day call reaches $1.065 \ (|10.732 - 11.797|)$, whereas the Black-Scholes price is identical to the MA(1)-type price for the $30$ 7-day call. A similar property can also be found in Table 2, where the Black-Scholes price undervalues the $30$ 364-day call by $2.592 \ (14.389 - 11.797)$ as $\beta = 0.75$.

Given the time to maturity and the autocorrelated parameter $\beta$, it is obvious that the differences between the in-the-money Black-Scholes prices and the MA(1)-type prices become large when the strike price increases. However, the pattern is not monotonic. For all panels illustrated in Tables 1 and 2, the differences eventually decline after the strike price $K$ is greater than $40$. To illustrate, in the case of $\beta = 0.75$, the difference in the Black-Scholes price and the MA(1)-type price is $0.49 \ (1.238 - 0.748)$ for the 7-day at-the-money call (as shown in Panel A of Table 2), although the difference is only $0.087 \ (0.091 - 0.004)$ and $0.002 \ (0.002 - 0.000)$ when the strike prices are $45$ and $50$, respectively.

Table 1 indicates that ignoring the impact of a negative autocorrelation induced by the MA(1)-type process can lead to large overpricing of MA(1)-type options. On the contrary, as shown in Table 2, ignoring the impact of a positive autocorrelation exhibited in stock returns can lead to very large underpricing of MA(1)-type options. Furthermore, it is also observed that the impact of autocorrelated stock returns on option prices is significant even when the autocorrelated parameter $\beta$ is small. For example, consider the $40$ 91-day call option. The corresponding option price will be overvalued by about 2.43% when $\beta = -0.025$ and will be undervalued by about 2.29% when $\beta = 0.025$. Accordingly, it is also found that
the MA(1)-type prices have some degree of asymmetry in the influence of $\beta$.

5 Conclusions

This paper proposes a method of valuing European options on an asset with autocorrelated returns. According to the common findings of empirical research studies, a continuous-time MA(1)-type process is introduced to describe the autocorrelated stock returns and the closed-form solution for option values under the MA(1)-type process is derived by using martingale pricing theory. The pricing formula of options on stocks with log-price dynamics given in (1) is similar to that of Black and Scholes except for the total volatility input. More specifically, the total volatility input of MA(1)-type options is the conditional standard deviation of continuously-compounded $\tau$-period returns, although the total volatility input of Black and Scholes is indeed the diffusion coefficient of a geometric Brownian motion times the square root of the option’s time to maturity $\tau$. Since the Black-Scholes formula is also applicable to pricing options on an asset with the trending O-U dynamics, the finding in this paper shows that the pricing formula for options on an asset with autocorrelated returns depends on the source of the autocorrelation. Furthermore, as shown in the numerical analyses, the impact of autocorrelation on the option values is significant and asymmetric, even when the autocorrelation of the underlying asset returns is weak.
Appendix

1. The proof of Lemma 1

To find the transformation from a $P$-Brownian motion to a $Q$-Brownian motion such that the dynamics of discounted stock prices, $\tilde{S}$, are a martingale under $Q$, it is useful to divide the option’s time to maturity $T - t_0$ into $i_T$ subintervals with the same length $h$ and the $(i_T + 1)$th subinterval with the length of $T - (t_0 + i_T h)$, where $i_T$ denotes the integer part of $(T - t_0)/h$.

To prove Lemma 1, the first time interval $[t_0, t_0 + h)$ is considered in the beginning. Based on (1) and (3), the discounted stock price $\tilde{S}_t$ satisfies
\[
\frac{d\tilde{S}_t}{\tilde{S}_t} = (\mu - r)dt + \sigma \beta dw_{t-h} + \sigma dW_t, \quad \forall \ t \in [t_0, t_0 + h). \tag{A.1}
\]
Note that for this time interval $t \in [t_0, t_0 + h)$, $dw_{t-h}$ is known under $\mathcal{F}_{t_0}$. It means that the non-random terms in the above equation are the drift term $(\mu - r)dt$ and $\sigma \beta dw_{t-h}$. By setting
\[
W_{t-t_0}^Q = W_{t_0} + \frac{(\mu - r)(t - t_0) + \int_{t_0}^{t} \sigma \beta dw_{s-h}}{\sigma}, \quad \forall \ t \in [t_0, t_0 + h),
\]
the dynamics of discounted stock prices under $Q$, $\tilde{S}_t^Q$, is
\[
\frac{d\tilde{S}_t^Q}{\tilde{S}_t^Q} = \sigma dW_t^Q, \quad \forall \ t \in [t_0, t_0 + h),
\]
which implies that $\tilde{S}$ is a martingale under $Q$ for this time interval. Thus, the transformation from $P$-Brownian to $Q$-Brownian for this interval is obtained as:
\[
dW_t^Q = dW_t + \frac{(\mu - r)}{\sigma} dt + \beta dw_{t-h}, \quad \forall \ t \in [t_0, t_0 + h). \tag{A.2}
\]
For the next time interval \( t \in [t_0 + h, t_0 + 2h] \), note that both \( dW_{t-h} \) and \( dW_t \) in Equation (1) are stochastic under \( \mathcal{F}_{t_0} \). Therefore, the dynamics of \( \tilde{S}_t \) are represented as:

\[
\frac{d\tilde{S}_t}{\tilde{S}_t} = (\mu - r)dt + \sigma dW_t + \sigma \beta dW_{t-h}, \quad \forall \ t \in [t_0 + h, t_0 + 2h). \tag{A.3}
\]

Based on (A.2) and the fact that \( dW_{t-h} \) for \( t \in [t_0 + h, t_0 + 2h) \) is identical to \( dW_t \) for \( t \in [t_0, t_0 + h) \), one can set

\[
W^Q_{t-(t_0+h)} = W_{t-(t_0+h)} + \frac{(1 - \beta)(\mu - r)[t - (t_0 + h)] - \int_{t_0+h}^{t} \sigma \beta^2 dw_{s-2h}}{\sigma}, \quad \forall \ t \in [t_0+h, t_0+2h),
\]

such that the process of \( \tilde{S}^Q_t \) is

\[
\frac{d\tilde{S}^Q_t}{\tilde{S}^Q_t} = \sigma dW^Q_t + \sigma \beta dW^Q_{t-h}, \quad \forall \ t \in [t_0 + h, t_0 + 2h).
\]

This implies that the discounted stock price \( \tilde{S} \) is a martingale under \( Q \). Accordingly, the transformation that makes \( \tilde{S} \) be a martingale under \( Q \) for this time interval is then

\[
dW^Q_t = dW_t + \frac{(1 - \beta)(\mu - r)}{\sigma} dt - \beta^2 dw_{t-2h}, \quad \forall \ t \in [t_0 + h, t_0 + 2h). \tag{A.4}
\]

The transformation from \( P \)-Brownian to \( Q \)-Brownian for the other time subintervals can be similarly obtained recursively and summarized as follows:

\[
dW^Q_z = dW_z + \sum_{j=0}^{i_z} \frac{(-\beta)^j(\mu - r)dz}{\sigma} + (-1)^{i_z} \beta^{i_z+1} dw_{z-(i_z+1)h}, \quad \forall \ z \in [t_0, T],
\]

where \( z = t, t - h, \) and \( i_z \) is the integer part of \( (z - t_0)/h \). The proof of Lemma 1 is now complete.
2. The dynamics of the stock price under measure $R$

Substituting (11) into (7), one can obtain the dynamics of the stock price under measure $R$, denoted as $S^R$, as follows:

\[
\frac{dS^R_t}{S^R_t} = rdt + 1_C \sigma^2 (1 + \beta) dt + 1_D \sigma^2 (1 + \beta)^2 dt + 1_E \sigma^2 [1 + \beta(1 + \beta)] dt \\
+ \sigma d(W^R_1 + \beta dW^R_2)_t,
\]

where

\[1_C \equiv 1\{t_0 \leq t < t_0 + h\}, \quad 1_D \equiv 1\{t_0 + h \leq t \leq T - h\}, \quad \text{and} \quad 1_E \equiv 1\{T - h < t \leq T\}\]

are indicator functions. By applying Itô’s lemma, $S^R_t$ can be solved as shown in Equation (12).
Bibliography


Footnote

1 The proof of this property appears in many mathematical finance books; for example, Klebaner (1998, p.258).

2 Please refer to Klebaner (1998, p. 242) for details.

3 Denote $\Phi_z$ as follows:

$$
\Phi_z = \begin{cases} 
-\sigma(1 + \beta), & \forall \ z \in [t_0, T-h]; \\
-\sigma, & \forall \ z \in (T-h, T].
\end{cases}
$$

Since $\Phi_z$ is bounded, the prerequisite for Girsanov’s theorem that $\Phi_z$ is a predictable process with the condition $\int_{t_0}^T \Phi_s^2 ds < \infty$ is satisfied.

4 Please refer to the Appendix for details.
## Table 1: Option Prices Under Negative Autocorrelated Stock Returns
(Daily Frequency)

<table>
<thead>
<tr>
<th>Strike</th>
<th>Black-Scholes Price</th>
<th>Panel A. Time-to-Maturity $T - t_0 = 7$ Days</th>
<th>Panel B. Time-to-Maturity $T - t_0 = 91$ Days</th>
<th>Panel C. Time-to-Maturity $T - t_0 = 182$ Days</th>
<th>Panel D. Time-to-Maturity $T - t_0 = 273$ Days</th>
<th>Panel E. Time-to-Maturity $T - t_0 = 364$ Days</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Price</td>
<td>Option Price Under Negative Autocorrelated Returns, with $\beta$ = -0.025</td>
<td>Option Price Under Negative Autocorrelated Returns, with $\beta$ = -0.050</td>
<td>Option Price Under Negative Autocorrelated Returns, with $\beta$ = -0.075</td>
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Note: This table compares call option prices on a hypothetical $40 stock under a geometric Brownian motion versus autocorrelated MA(1)-type stock returns. The parameter used for the coefficient of $dW_t$, i.e., $\sigma$, is 1.75 percent for daily returns, and the daily continuously-compounded risk-free rate is log(1.025)/365.
Table 2: Option Prices Under Positive Autocorrelated Stock Returns  
(Daily Frequency)

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Panel A. Time-to-Maturity $T - t_0 = 7$ Days

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<th>Panel D</th>
<th>Panel E</th>
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Panel B. Time-to-Maturity $T - t_0 = 91$ Days

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Panel C. Time-to-Maturity $T - t_0 = 182$ Days

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<tr>
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Panel D. Time-to-Maturity $T - t_0 = 273$ Days

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Panel E. Time-to-Maturity $T - t_0 = 364$ Days

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Note: This table compares call option prices on a hypothetical $40 stock under a geometric Brownian motion versus autocorrelated MA(1)-type stock returns. The parameter used for the coefficient of $dW_t$, i.e., $\sigma$, is 1.75 percent for daily returns, and the daily continuously-compounded risk-free rate is $\log(1.025)/365$. 
