The **VALUATION OF RESET OPTIONS WITH MULTIPLE STRIKE RESETS AND RESET DATES**

SZU-LANG LIAO*
Department of Money and Banking, National Chengchi University
64, Chih-nan Rd.,Sec. 2, Taipei 116, Taiwan
Tel : 886-2-29393091 ext 81251 Fax : 886-2-29398004
Email Address: liaosl@nccu.edu.tw

CHOU-WEN WANG
Department of Money and Banking, National Chengchi University
64, Chih-nan Rd.,Sec. 2, Taipei 116, Taiwan
Tel : 886-2-29393091 ext 81251 Fax : 886-2-29398004
Email Address: g8352504@m0.nccu.edu.tw

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This paper makes two contributions to the literature. The first contribution is to provide the closed-form pricing formulas of reset options with \( m \) strike resets and \( n \) pre-decided reset dates. The exact closed-form pricing formulas of reset options with \( m \) strike resets and continuous reset period are also derived. The second contribution is the finding that the reset options not only have the phenomena of Delta jump and Gamma jump across reset dates, but also have the properties of Delta waviness and Gamma waviness, especially nearby the time before reset dates. Furthermore, Delta and Gamma can be negative when stock price is near the strike resets at the times close to reset dates.
INTRODUCTION

Path-dependent options, whose payoffs are influenced by the path of the prices of underlying assets, have become increasingly popular in recent years. One of the path-dependent options is look-back option whose payoff depends, in particular, on the minimum or maximum price of the underlying asset during the option’s lifetime. There is another kind of path-dependent option known as reset option. Unlike the look-back option, the strike price of reset option will be reset to a new strike price only on the pre-specified reset dates if the price of underlying assets is lower than one of the strike resets.

Reset options have been issued in practice for many years. The Chicago Board Options Exchange (CBOE) and the New York Stock Exchange (NYSE) both introduced S&P 500 index put warrants with three-month reset period in late 1996. Morgan Stanley issued a reset warrant with initial strike price $44.73 in July 1997. The strike price will be adjusted to $39.76 on August 5, 1997 if the price of underlying asset falls below $39.76. A more recent example comes from Taiwan, Grand Cathay Securities had six reset options listed in Taiwan Stock Exchange (TSE) (Codes in TSE are 0517, 0522, 0523, 0527, 0528, and 0538) from 1998 to 1999. Most reset options, including all of the reset options listed in TSE, are options with multiple strike resets and reset dates. For example, the reset condition of 0522 of TSE is that the strike price will be adjusted if the six-day average closed price of 2323 in TSE falls below 98%, 96%, 94%, 92%, 90% of initial strike price $81 during the first three months after the issue of warrant.

Because the reset warrants are new derivative products in financial markets, few studies have been done on their pricing problems. Gray and Whaley (1997) examined the pricing of put warrant with periodic reset and the warrant’s risk characteristics. They further provided a closed-from solution for reset options with single reset date in a latter
paper (Gray and Whaley (1999)). Cheng and Zhang (2000) studied the reset options that the strike price will be reset to the prevailing stock price if the option is out of money. A closed-form pricing formula in terms of multivariate normal distribution is derived under the risk-neutral framework. However, the reset conditions of reset options investigated by Cheng and Zhang (2000) are not the general cases of reset products in practice. Let the underlying asset price at time $t$ be denoted by $S(t)$. The terminal payoff of reset option with $n$ reset dates and initial strike price $K_0$, which was studied by Cheng and Zhang (2000), is as follows:

$$C(T) = \max \left[ S(T) - \min \left[ K_0, S(t_1), ..., S(t_n) \right], 0 \right]. \quad (1)$$

In practice, however, the terminal payoff of reset option is more often set as

$$C(T) = \max \left[ S(T) - K^*, 0 \right] = \left[ S(T) - K^* \right]^+, \quad (2)$$

where

$$K^* = \begin{cases} 
K_0 & \text{if } \min \left[ S(t_1), ..., S(t_n) \right] \geq D_i, \\
K_i & \text{if } D_i > \min \left[ S(t_1), ..., S(t_n) \right] \geq D_{i+1}, \quad i = 1, ..., m - 1, \\
K_m & \text{if } D_m > \min \left[ S(t_1), ..., S(t_n) \right], 
\end{cases} \quad (3)$$

and $K_i, \quad i = 1, ..., m$, are the reset strike prices; $D_i, \quad i = 1, ..., m$, are the strike resets.

Our first contribution in this article is to derive the exact closed-form solution for reset options with $m$ strike resets and $n$ pre-decided reset dates, as specified in (2) and (3), under the risk-neutral framework. Furthermore, we also provide the closed-form solution for reset options with $m$ strike resets and continuous reset dates, which is the limiting case of the former.

Some previous studies, such as Cheng and Zhang (2000), have pointed out the phenomenon of Delta jump across reset dates. The second contribution of this paper is the finding that, in addition to Delta jump, a reset option with $m$ strike resets also has the phenomena of Gamma jump, Delta waviness and Gamma waviness as well. The
waviness of delta and gamma means that the delta and gamma of reset options will oscillate when the stock price passes across the strike resets. When the time is approaching reset dates and the stock price is near the strike resets, delta and gamma may change their values from positive to negative. The phenomena of Delta jump and Gamma jump near reset time as well as the properties of Delta waviness and Gamma waviness will make the risk management more difficult.

**PRICING RESET OPTIONS WITH $m$ STRIKE RESETS AND $n$ RESET DATES**

We assume the dynamics of underlying asset price $S(t)$ are described by the following stochastic differential equation:

$$dS(t) = uS(t)dt + \sigma S(t)dW_t,$$

where $u$ and $\sigma > 0$ are constants, and $W_t$ is a one-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, P)$. The money market account, $B(t)$, corresponds to the wealth accumulated by an initial $1$ investment at spot interest rate $r$ in each subsequent period. Therefore,

$$dB(t) = rB(t)dt,$$

or equivalently,

$$B(T) = B(t)e^{rt(1-t)}.$$

Let $Q$ be the spot martingale measure with Radon-Nikodym derivative

$$\frac{dQ}{dP} = \exp\left(\frac{r - u}{\sigma} W_T - \frac{1}{2}\frac{r - u}{\sigma}^2 T\right).$$

Under the spot martingale measure or risk neutral probability measure $Q$, the dynamics of underlying asset price $S(t)$ become

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q_t,$$
where the process $W_t^Q$ is defined by
\[
dW_t^Q = dW_t - \frac{r - \mu}{\sigma} dt.
\] (9)

In view of (2) and (3), the payoff at expiry of reset option with $m$ strike resets and $n$ pre-decided reset dates can be written as
\[
C(T) = \left[ S(T) - K_0 \right]^+ I_{\left\{ \min_{1 \leq j \leq n} S(t_j) \geq D_1 \right\}} + \\
\left[ S(T) - K_1 \right]^+ \left\{ I_{\left\{ \min_{1 \leq j \leq n} S(t_j) \geq D_2 \right\}} - I_{\left\{ \min_{1 \leq j \leq n} S(t_j) \geq D_1 \right\}} \right\} + \ldots \\
+ \left[ S(T) - K_{m-1} \right]^+ \left\{ I_{\left\{ \min_{1 \leq j \leq n} S(t_j) \geq D_m \right\}} - I_{\left\{ \min_{1 \leq j \leq n} S(t_j) \geq D_{m-1} \right\}} \right\} \\
+ \left[ S(T) - K_m \right]^+ \left\{ 1 - I_{\left\{ \min_{1 \leq j \leq n} S(t_j) \geq D_m \right\}} \right\},
\] (10)
where $I(\cdot)$’s are indicator functions. Under the risk neutral probability measure $Q$, the arbitrage-free price of reset option $C(t)$ at time $t$ is
\[
C(t) = e^{-r(T-t)} E_Q \left[ C(T) \mid F_t \right]
\]
\[
= e^{-r(T-t)} \sum_{l=1}^{m} E_Q \left\{ \left[ S(T) - K_{l-1} \right]^+ \left\{ I_{\left\{ \min_{1 \leq j \leq n} S(t_j) \geq D_l \right\}} \right\} \mid F_t \right\} \\
- e^{-r(T-t)} \sum_{l=1}^{m} E_Q \left\{ \left[ S(T) - K_l \right]^+ \left\{ I_{\left\{ \min_{1 \leq j \leq n} S(t_j) \geq D_l \right\}} \right\} \mid F_t \right\} \\
+ e^{-r(T-t)} E_Q \left\{ \left[ S(T) - K_m \right]^+ \mid F_t \right\}
\] (11)

From (11), we know that the key of solution is to compute the following expression:
\[
e^{-r(T-t)} E_Q \left\{ \left[ S(T) - K_h \right]^+ \left\{ I_{\left\{ \min_{1 \leq j \leq n} S(t_j) \geq D_l \right\}} \right\} \mid F_t \right\}.
\] (12)

We present the result in the following Theorem.

THEOREM: The explicit solution to (12) is as follows:
\[ e^{-r(T-t)} E_Q \left\{ \left[ S(T) - K_h \right] I \left( \min_{1 \leq j \leq n} S(t_j) \geq D_i \right) \bigg| F_t \right\} \]

\[ = \sum_{g=1}^{n} \left[ S(t) N_{n+1}(D_g^{i,h}, \Sigma_g) - K_h e^{-r(T-t)} N_{n+1}(\hat{D}_g^{i,h}, \Sigma_g) \right], \tag{13} \]

where \( N_{n+1}(\cdot; \Sigma) \) is the cumulative probability of \((n+1)\)-dimensional multivariate normal distribution with mean vector 0 and covariance matrix \( \Sigma \). For \( i, h = 1, \ldots, m \), the parameters in (13) are defined as follows:

\[
D^{i,h} = \begin{bmatrix}
d_{i,1} & e_{i,2} & \cdots & e_{i,n} & y_h \\
e_{2,1} & d_{i,2} & \cdots & e_{i,n} & y_h \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
e_{n,1} & e_{n,2} & \cdots & d_{i,n} & y_h
\end{bmatrix} \tag{14}
\]

and \( D_j^{i,h} \) stands for the \( j^{th} \) row of \( D^{i,h} \):

\[
d_{i,j} = \frac{\ln(S(t)/D_i) + (r + \frac{1}{2} \sigma^2)(t_j - t)}{\sigma \sqrt{t_j - t}} \tag{15}
\]

\[
e_{i,j} = \frac{(r + \frac{1}{2} \sigma^2)(t_j - t_i)}{\sigma \sqrt{|t_j - t_i|}} \tag{16}
\]

\[
y_h = \frac{\ln(K_h/S(t)) + (r + \frac{1}{2} \sigma^2)(T - t)}{\sigma \sqrt{T - t}} \tag{17}
\]

\( \hat{D}^{i,h} \) is similarly defined as \( D^{i,h} \) with the parameters \( d_{i,j}, e_{i,j}, \) and \( y_h \) replaced by \( \hat{d}_{i,j}, \hat{e}_{i,j}, \) and \( \hat{y}_h \), respectively:

\[
\hat{d}_{i,j} = d_{i,j} - \sigma \sqrt{t_j - t} \tag{18}
\]

\[
\hat{e}_{i,j} = \frac{(r - \frac{1}{2} \sigma^2)(t_j - t_i)}{\sigma \sqrt{|t_j - t_i|}} \tag{19}
\]

\[
\hat{y}_h = y_h - \sigma \sqrt{T - t} \tag{20}
\]
and the correlation matrix

$$\Sigma_g = \left\{ \rho_{ij}^g \right\}_{(n+1) \times (n+1)} \quad i, j = 1, \ldots, n + 1$$  \hspace{1cm} (21)

where $\rho_{ij}^g$'s are given by

$$\rho_{ij}^g = \begin{cases} 
1, & i = j \\
\sqrt{\frac{t_i - t_j}{t_i - t_j}}, & 1 \leq i < j \leq g - 1 \text{ or } g + 1 \leq i < j \leq n, \\
-\sqrt{\frac{t_i - t_j}{t_i - t_j}}, & 1 \leq i \leq g - 1, j = g, \\
-\sqrt{\frac{t_i - t_j}{T - t}}, & 1 \leq i \leq g - 1, j = n + 1, \\
\sqrt{\frac{t_i - t_j}{T - t}}, & g + 1 \leq i \leq n, j = n + 1, \\
\sqrt{\frac{t_i - t_j}{T - t}}, & i = g, j = n + 1, \\
0, & \text{otherwise.} \end{cases}$$  \hspace{1cm} (22)

We prove the Theorem in Appendix A.

Accordingly, the closed-form solution for reset option with $m$ strike resets and $n$ pre-decided reset dates $C(t)$ is

$$C(t) = S(t) \left[ N(y_m) + \sum_{i=1}^{m} \sum_{g=1}^{n} \left( N_{n+1}(D_{g, i}^{l, i-1}; \Sigma_g) - N_{n+1}(D_{g, i}^{l, i}; \Sigma_g) \right) \right]$$

$$- \sum_{i=1}^{m-1} K_i e^{-r(T-t)} \left[ \sum_{g=1}^{n} \left( N_{n+1}(\hat{D}_{g, i}^{l, i}; \Sigma_g) - N_{n+1}(\hat{D}_{g, i}^{l, i}; \Sigma_g) \right) \right]$$

$$- K_0 e^{-r(T-t)} \sum_{g=1}^{n} N_{n+1}(\hat{D}_{g}^{l, 0}; \Sigma_g) - K_m e^{-r(T-t)} \left[ N(\hat{y}_m) - \sum_{g=1}^{n} N_{n+1}(\hat{D}_{g}^{m, m}; \Sigma_g) \right]$$  \hspace{1cm} (23)

where $N(\cdot)$ is the cumulative probability of standard normal distribution.

In view of (23), we can replicate the reset option by purchasing $\Delta$ shares of stock

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1 Here we define $T = t_{n+1}$.\'
at price $S(t)$ and borrowing $M$ dollars. The amount $\Delta$ and $M$ are as follows:

$$\Delta = N(y_m) + \sum_{i=1}^{m} \sum_{g=1}^{n} \left[ N_{n+1}(D^{i,l-1}_g, \Sigma_g) + N_{n+1}(D^{l,l}_g, \Sigma_g) \right]$$  \hspace{1cm} (24)

$$M = \sum_{i=1}^{m-1} K_i e^{-r(T-t)} \left\{ \sum_{g=1}^{n} \left[ N_{n+1}(\hat{D}^{i,l-1}_g, \Sigma_g) - N_{n+1}(\hat{D}^{l,l}_g, \Sigma_g) \right] \right\}$$

$$+ K_0 e^{-r(T-t)} \sum_{g=1}^{n} N_{n+1}(\hat{D}^{1,0}_g, \Sigma_g) + K_m e^{-r(T-t)} \left[ N(\hat{y}_m) - \sum_{g=1}^{n} N_{n+1}(\hat{D}^{m,m}_g, \Sigma_g) \right]$$  \hspace{1cm} (25)

Similar to the closed-form valuations of exotic options such as options on the maximum or minimum of several assets (Johnson (1987)), discrete partial barrier options (Heynen and Kat (1996)), reset options (Cheng and Zhang (2000)) or economic models with limited dependent variables, including multinomial probit, panel studies, spatial analysis, and time series analysis, the closed-form solutions for reset options involve the multivariate normal distribution functions.

Among the methods of evaluating multivariate normal cumulative probabilities, as Gollwitzer and Rackwitz (1987), Deàk (1988), and Vijverberg (1997) had pointed out, Monte Carlo simulator methods seem to be the most promising for higher-order probabilities, preferable over analytical approximations or numerical integration methods. Hajivassiliou, McFadden and Ruud (1996) surveyed eleven Monte Carlo techniques of evaluating multivariate normal probabilities, they found that Geweke-Hajivassiliou-Keane (GHK) simulator is overall the most reliable method. Consequently, for the closed-form solution for reset options with large number of reset dates, we suggest GHK simulator for computing the multivariate normal cumulative probabilities.\(^2\)

In practice, reset derivatives are usually related to the arithmetic averages of stock

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\(^2\) The computer codes in Matlab for computing the price of reset options are available in the authors’ websites.
prices in most financial markets. Consequently, we denote $A(t_j)$ as the arithmetic average of stock prices at time $t_j$. Then, for an arithmetic average reset option with $m$ strike resets and $n$ pre-decided reset dates, the terminal payoff becomes

$$C(T) = \left[ S(T) - K^* \right],$$

where

$$K^* = \begin{cases} 
K_0 & \text{if } \min[A(t_1), \ldots, A(t_n)] \geq D_i \\
K_i & \text{if } D_i > \min[A(t_1), \ldots, A(t_n)] \geq D_{i+1}, \quad i = 1, \ldots, m-1, \\
K_m & \text{if } D_m > \min[A(t_1), \ldots, A(t_n)] 
\end{cases}$$

Since the sum of lognormal variables is not lognormal and there is no recognizable probability distribution for it, there are no closed-form pricing formulas for the options based on the arithmetic average of asset values.

However, we can derive an approximated closed-form formula for the arithmetic average reset options by assuming the arithmetic averages $A(t_j)$ are approximately lognormally distributed. Using Wilkinson approximation, which is also used by Levy (1992) in pricing Asian options, we may estimate the mean and standard deviation of $\log A(t_j)$ through the true first two moments of $A(t_j)$. Then, following the similar procedure in Appendix A, we can derive the closed-form formulas straightforwardly.\(^3\)

**ANALYSES OF THE RESET OPTIONS**

**Characteristics of Rest Options**

First, we discuss some properties of reset options. Consider a one-year maturity reset option with initial strike price at 100. The strike price will be adjusted if the closed price of underlying stock falls below 90%, 80% of the initial strike price. We will compare the prices of the reset options with 2 strike resets and one, two and three reset dates to the

\(^3\) The approximated closed-form formulas of arithmetic average reset options are available upon request.
plain vanilla call option. The results are presented in Table I.

From Table I, we can see that some characteristics of reset options are similar to the standard European call option. For example, the values of reset options are increasing functions of stock price, risk-free interest rate, and the volatility of stock returns. In addition, there are four properties that uniquely exist in reset options. First, the values of reset options are increasing with the number of reset dates. Second, under the same strike resets $D_j$, lower reset strike prices $K_j$ will result in higher values of reset options. Third, due to the more protection toward the holders of reset options, the values of reset options are always greater than that of standard European call option. Finally, in the cases of higher values of stock price than the strike resets and smaller volatility of stock returns, the difference between the prices of reset options and plain vanilla call options is insignificant. Take the stock price 115 and the volatility of stock returns 10% as an example. The prices of them are almost the same.

**Reset Options with Continuous Reset Dates**

When $n$ approaches infinity with a remaining time to maturity $T - t$, the set of discrete reset dates become a continuous reset period. The terminal payoff of reset option with continuous reset periods is as follows:

$$C(T) = C_T^m + \sum_{i=1}^{m} C_i^{j=1} I(Min S(t) \geq D_j) - \sum_{i=1}^{m} C_i^{j} I(Min S(t) \geq D_j),$$

(28)

where $C_T^{j} = (S(T) - K_j)^+$. In views of (28), we can replicate the reset option with the following strategy:

1. Purchase one unit of European call option with strike price $K_m$.
2. Purchase one unit of European down and out call option with strike price $K_{i-1}$ and barrier $D_i$, for each $i = 1, \ldots, m$. 

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TABLE 1

Prices of Plain Vanilla Call Option \((n = 0)\) and Reset Options with 2 Strike Resets and Multiple Reset Dates \((n = 1, 2, 3)\)

<table>
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<th>(\sigma)</th>
<th>(S(t))</th>
<th>((K_1, K_2))</th>
<th>(n)</th>
<th>(\sigma)</th>
<th>(S(t))</th>
<th>((K_1, K_2))</th>
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Here, \(S(t) = 100, K_2 = 100, D_1 = 90, D_2 = 80, t = 0, T = 1\). The reset dates are the last day of each month. \(n\) represents the number of reset dates, for example, \(n = 3\) means that the reset dates are 1/12, 2/12 and 3/12.
3. Short sell one unit of European down and out call option with strike price $K_i$ and barrier $D_j$, for each $i = 1, \ldots, m$.

Consequently, we can derive the pricing formulas of reset options with continuous reset period by discovering the prices of down and out call options. Based on the closed-form solutions of European single-barrier options provided by Rubinstein and Reiner (1991), we have

$$e^{-r(T-t)}E_Q \left\{ \left[ S(T) - K_j \right]^+ I \left( \min_{j \leq u \leq T} S(u) \geq D_{j+1} \right) \right\}$$

$$= \left\{ S(t) \left[ N(y_j) - \left( \frac{D_{j+1}}{S(t)} \right)^{2(r+0.5\sigma^2)} N(f_{j+1,j}^{m,j}) \right] + K_j e^{-r(T-t)} \left[ N(\tilde{y}_j) - \left( \frac{D_{j+1}}{S(t)} \right)^{2(r-0.5\sigma^2)} N(f_{j+1,j}^{m,j}) \right] \right\}$$

where

$$f_{1/2}^{i,j} = \frac{\ln\left( \frac{D_i^2}{S(t)K_j} \right) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Therefore, the price of reset option with continuous reset period is

$$C(t) = S(t) \left\{ N(y_0) + \sum_{j=1}^{m} \left( \frac{D_j}{S(t)} \right)^{2(r+0.5\sigma^2)} \left[ N(f_{1,j}^{1,j}) + N(f_{1,j}^{1,i-1}) \right] \right\}$$

$$- \sum_{j=1}^{m-1} K_j e^{-r(T-t)} \left[ \left( \frac{D_j}{S(t)} \right)^{2(r+0.5\sigma^2)} N(f_{1,j}^{1,j}) + \left( \frac{D_{j+1}}{S(t)} \right)^{2(r-0.5\sigma^2)} N(f_{1,j+1}^{1,j}) \right]$$

---

\[ + K_0 e^{-r(T-t)} \left( \frac{D_1}{S(t)} \right) \frac{2(r-0.5\sigma^2)}{\sigma^2} N(f_{2,0}^{1,0}) - K_m e^{-r(T-t)} \left[ N(\hat{y}_m) + \frac{D_m}{S(t)} \frac{2(r-0.5\sigma^2)}{\sigma^2} N(f_{2,m}^{m,m}) \right] \] (31)

**Delta Jump and Gamma Jump**

We now consider some important properties of reset options, such as Delta jump and Gamma jump. When the reset options are issued, the issuers must hedge the risk exposure induced by reset options. We provide the delta and gamma of reset options in Appendix B.

To describe the phenomena of Delta jump and Gamma jump, without loss of generality, we simplify the reset options with only one reset date. Let us define the following expressions:

\[ X(i, j) = \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{d_{j,1}^2}{2} \right) N(G_{i,j}) \] (32)

\[ Y(i, j) = \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{y_i^2}{2} \right) N(Z_{i,j}) \] (33)

\[ DX(i, j) = \frac{\exp(-d_{j,1}^2)}{\alpha S(t) \sqrt{2\pi}} \left[ \frac{N(G_{i,j})d_{j,1}}{\sqrt{t_i-t}} + \frac{\exp(-G_{i,j}^2)}{\sqrt{2\pi}(T-t)(1-\rho^2)} - \frac{\rho \exp(-G_{i,j}^2)}{\sqrt{2\pi}(t_i-t)(1-\rho^2)} \right] \] (34)

\[ DY(i, j) = \frac{\exp(-y_i^2)}{\alpha S(t) \sqrt{2\pi}} \left[ \frac{N(Z_{i,j})y_i}{\sqrt{t_i-t}} + \frac{\exp(-Z_{i,j}^2)}{\sqrt{2\pi}(t_i-t)(1-\rho^2)} - \frac{\rho \exp(-Z_{i,j}^2)}{\sqrt{2\pi}(t_i-t)(1-\rho^2)} \right] \] (35)

where \( G_{i,j} = \frac{y_i - \rho d_{j,1}}{\sqrt{1-\rho^2}}, \quad Z_{i,j} = \frac{d_{j,1} - \rho y_i}{\sqrt{1-\rho^2}}; \quad \hat{X}, \quad \hat{Y}, \quad D\hat{X}, \quad \text{and} \quad D\hat{Y} \) are the expressions with \( d_{i,j} \) and \( y_i \) replaced by \( \hat{d}_{i,j} \) and \( \hat{y}_i \), respectively. Thus the delta and gamma of reset options with one reset date are as follows:

\[ \text{Delta}(t, S(t)) = \]
\[
N(y_m) + \sum_{i=1}^{m} \left\{ N_2(d_{t,i}, y_{t-1}, \Sigma_t) - N_3(d_{t,i}, y_{t+1}, \Sigma_t) \right\} + \frac{1}{\sigma \sqrt{T-t}} \left[ X(l-1,l) - X(l,l) \right] + \frac{1}{\sigma \sqrt{T-t}} \left[ Y(l-1,l) - Y(l,l) \right]
\]

\[
- \sum_{i=1}^{m-1} K_i e^{-r(T-t)} \left\{ \frac{1}{\sigma \sqrt{T-t}} \left[ \tilde{X}(l,i+1) - \tilde{X}(l,i) \right] + \frac{1}{\sigma \sqrt{T-t}} \left[ \tilde{Y}(l,i+1) - \tilde{Y}(l,i) \right] \right\}
\]

\[
- \frac{K_i e^{-r(T-t)}}{S(t)} \left[ \frac{1}{\sigma \sqrt{T-t}} \tilde{X}(0,1) + \frac{1}{\sigma \sqrt{T-t}} \tilde{Y}(0,1) \right]
\]

\[
+ \frac{K_i e^{-r(T-t)}}{S(t)} \left[ \frac{1}{\sigma \sqrt{T-t}} \tilde{X}(m,m) + \frac{1}{\sigma \sqrt{T-t}} \tilde{Y}(m,m) \right]
\]

\[
\text{Gamma}(t,S(t)) = \exp\left( -\frac{y_m^2}{2} \right) \frac{1}{S(t) \sqrt{2\pi(T-t)}} + \sum_{i=1}^{m} \frac{1}{\sigma \sqrt{T-t}} \left[ X(l-1,l) - X(l,l) \right] + \frac{1}{\sigma \sqrt{T-t}} \left[ Y(l-1,l) - Y(l,l) \right]
\]

\[
+ \sum_{i=1}^{m} \frac{1}{\sigma S(t)^2} \left[ DX(l-1,l) - DX(l,l) \right] + \frac{1}{\sigma S(t)^2} \left[ DY(l-1,l) - DY(l,l) \right]
\]

\[
+ \sum_{i=1}^{m} \frac{1}{\sigma S(t)^2} \left[ DX(l,i+1) - DX(l,i) \right] + \frac{1}{\sigma S(t)^2} \left[ DY(l,i+1) - DY(l,i) \right]
\]

\[
- \frac{S(t)}{\sqrt{T-t}} \left[ \tilde{D}X(l,i+1) - \tilde{D}X(l,i) \right] - \frac{S(t)}{\sqrt{T-t}} \left[ \tilde{D}Y(l,i+1) - \tilde{D}Y(l,i) \right]
\]

\[
+ \frac{K_i e^{-r(T-t)}}{\sigma S(t)^2} \left[ \frac{1}{\sqrt{T-t}} \tilde{X}(0,1) - \frac{S(t)}{\sqrt{T-t}} \tilde{D}X(0,1) \right] - \frac{K_i e^{-r(T-t)}}{\sigma S(t)^2} \left[ \frac{1}{\sqrt{T-t}} \tilde{Y}(0,1) - \frac{S(t)}{\sqrt{T-t}} \tilde{D}Y(0,1) \right]
\]

\[
- \frac{K_i e^{-r(T-t)}}{\sigma S(t)^2} \left[ \frac{1}{\sqrt{T-t}} \tilde{X}(m,m) - \frac{S(t)}{\sqrt{T-t}} \tilde{D}X(m,m) \right] - \frac{K_i e^{-r(T-t)}}{\sigma S(t)^2} \left[ \frac{1}{\sqrt{T-t}} \tilde{Y}(m,m) - \frac{S(t)}{\sqrt{T-t}} \tilde{D}Y(m,m) \right]
\]

where \( t \) is the reset date.

When \( t \to t_1 \), then \( N_2(d_{i,j}, y_{t-1}, \Sigma_t) \to N(y_{t-1}), \ X(i, j) \to 0, \ DX(i, j) \to 0, \ Y(i, j) \to 0 \)
\[ \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y_j^2}{2}\right) \bigg|_{t=t_1}, \quad \text{and} \quad Dy(i, j) \rightarrow \frac{1}{\sigma S(t)\sqrt{2\pi(T-t)}} \exp\left(\frac{-y_j^2}{2}\right) \bigg|_{t=t_1}. \]

Consequently, the Delta and Gamma at time \( t_1 \) are as follows:

\[ \text{Delta}(t_1, S(t_1)) = N(y_0) \bigg|_{t=t_1} \tag{38} \]

\[ \Gamma(t_1, S(t_1)) = \frac{\exp(-\frac{y_0^2}{2})}{S(t)\sigma \sqrt{2\pi(T-t)}} \bigg|_{t=t_1} \tag{39} \]

However, the delta and gamma at \( t > t_1 \) are given by the following expressions:

\[ \text{Delta}(t, S(t)) = N(y_0) I[S(t_1) \geq D_1] + \sum_{g=1}^{m-1} N(y_g) I[D_g > S(t_1) \geq D_{g+1}] + N(y_m) I[D_m > S(t_1)] \tag{40} \]

\[ \Gamma(t, S(t)) = \frac{\exp(-\frac{y_0^2}{2})}{S(t)\sigma \sqrt{2\pi(T-t)}} I[S(t_1) \geq D_1] \]

\[ + \sum_{g=1}^{m-1} \frac{\exp(-\frac{y_g^2}{2})}{S(t)\sigma \sqrt{2\pi(T-t)}} I[D_g > S(t_1) \geq D_{g+1}] + \frac{\exp(-\frac{y_m^2}{2})}{S(t)\sigma \sqrt{2\pi(T-t)}} I[D_m > S(t_1)] \tag{41} \]

From (38) to (41), we can see that the Delta and Gamma at \( t_1 \) are continuous only when the condition \( S(t_1) \geq D_1 \) holds. Therefore, Delta jump and Gamma jump exist when the stock price at \( t_1 \) is below \( D_1 \). In other words, we should carefully implement the Delta and Gamma hedges at the moments of reset dates under the case that the stock price is below the highest strike reset.

**Delta Waviness and Gamma Waviness**

In addition to the properties of Delta jump and Gamma jump at the moments of reset dates, there exist the phenomena of Delta waviness and Gamma waviness before the reset dates, especially nearby the reset dates. Consider the following example. The stock price is currently $100, and the strike price of reset option will be adjusted if the stock
price falls below 80%, 70%, 60%, 50%, and 40% of initial strike price $100 at three months later. Assume the risk-free interest rate is 5% and the volatility of stock returns is 30%. We illustrate the properties of Delta waviness and Gamma waviness in Figure 1 and Figure 2, respectively. As shown in the Figures, unlike the Delta and Gamma of the plain vanilla call options which are definitely non-negative, the Delta and Gamma of reset options will fluctuate dramatically and can be negative as the time approaches reset dates. When the stock prices are away from the neighborhoods of strike resets, the behaviors of Delta and Gamma are the same as that of plain vanilla call options. However, if the stock prices are near strike resets, the Delta and Gamma will oscillate. The phenomena are more significant when the time approaches reset dates. From Figure 1, if the time approaches reset dates, the Delta is a local minimum when the stock price touches strike reset, but the Delta is a local maximum when the stock price is at about the middle of two adjacent strike resets. The dramatic change of Delta between two adjacent strike resets also increases the difficulty of risk management. The waviness of Delta and Gamma are as important as Delta jump and Gamma jump in hedging reset options.

**CONCLUSION**

We have provided the closed-form pricing formula for reset option with \( m \) strike resets and \( n \) pre-decided reset dates. In addition to the Delta jump and Gamma jump across reset dates, we have also discovered the phenomena of Delta waviness and Gamma waviness nearby the times of reset dates. For future research, it is interesting to investigate the hedging strategies of the reset options due to the phenomena of Delta jump and Delta waviness across reset dates.
FIGURE 1. Delta of Reset Option with Five Strike Resets. Here, $S(t) = 100$, $K_o = 100$, $[K_1, ..., K_5] = [80, 70, 60, 50, 40]$, $[D_1, ..., D_5] = [80, 70, 60, 50, 40]$, $r = 0.05$, and $v = 0.3$. Unlike the Delta of the plain vanilla call option which is definitely non-negative, the Delta of reset call option will fluctuate dramatically and may be negative as the time approaches reset dates. The Deltas are local minimums when the stock price touches strike resets, but the Deltas are local maximums when the stock price is at about the middle of two adjacent strike resets. The phenomenon of Delta waviness is more significant as the time approaches reset dates.
FIGURE 2. Gamma of Reset Option with Five Strike Resets. Here, $S(t) = 100$, $K_0 = 100$, $[K_1, ..., K_5] = [80, 70, 60, 50, 40]$, $[D_1, ..., D_5] = [80, 70, 60, 50, 40]$, $r = 0.05$, and $v = 0.3$. When the stock price is away from the neighborhoods of strike resets, the behavior of Gamma is the same as that of plain vanilla call option. However, if the stock price is near strike resets, the Gamma oscillates across the strike resets. The phenomenon of Gamma waviness is more significant when the time approaches reset dates.
APPENDIX A

Proof of the Theorem

To carry out the proof of Theorem 1, we divide (12) into two parts:

\[
e^{-r(T-t)}E_Q \left\{ \left[ S(T) - K_h \right]^+ I\left( \min_{j=1,n} S(t_j) \geq D_i \right) \bigg| F_i \right\} = A - B \quad (A.1)
\]

where

\[
A = e^{-r(T-t)} \sum_{g=1}^{n} E_Q \left\{ S(T)I\left( S(t_g) \geq D_i, S(T) \geq K_h, S(t_j) \geq S(t_g), j \neq g, j = 1,\ldots,n \right) \bigg| F_i \right\} \quad (A.2)
\]

\[
B = K_h e^{-r(T-t)} \sum_{g=1}^{n} E_Q \left\{ I\left( S(t_g) \geq D_i, S(T) \geq K_h, S(t_j) \geq S(t_g), j \neq g, j = 1,\ldots,n \right) \bigg| F_i \right\} \quad (A.3)
\]

Under the spot martingale measure \( Q \), the stock price at time \( t \) equals

\[
S(t_j) = S(t) \exp \left[ (r - \frac{1}{2} \sigma^2)(t_j - t) + \sigma (W^Q_t - W^R_t) \right] \quad (A.4)
\]

It is convenient to introduce an auxiliary probability measure \( P_R \) on \((\Omega, F)\) by setting its Radon-Nikodym derivative as follows:

\[
\frac{dP_R}{dQ} = \exp \left[ \sigma W^Q_t - \frac{1}{2} \sigma^2 t \right] \quad (A.5)
\]

By Girsanov’s theorem, \( W^R_t \), defined by

\[
dW^R_t = dW^Q_t - \sigma dt \quad (A.6)
\]

is a standard Brownian motion under the measure \( P_R \). Then we can rewrite (A.2) as follows:

\[
A = S(t) \sum_{g=1}^{n} P_R \left\{ \left( S(t_g) \geq D_i, S(T) \geq K_h, S(t_j) \geq S(t_g), j \neq g, j = 1,\ldots,n \right) \bigg| F_i \right\}
\]

\[
\geq S(t) \sum_{g=1}^{n} P_R \left[ \ln S(t_g) - \ln D_i + \left( r + \frac{1}{2} \sigma^2 \right) (t_g - t) - \frac{1}{\sigma \sqrt{t_g - t}} \left( W^R_{t_g} - W^R_t \right) \right] \geq \frac{-(W^R_{t_g} - W^R_t)}{\sqrt{t_g - t}} \quad (A.7)
\]

\footnote{We can represent the minimum of several assets as the expression in (A.2), for details, see Johnson (1987).}
\[
\frac{\ln S(T) - \ln K_t + (r + \frac{1}{2} \sigma^2)(T-t)}{\sigma \sqrt{T-t}} \geq \frac{-(W_t^R - W_t^R)}{\sqrt{T-t}}, \quad (A.7)
\]

Here we use the fact that \( W_t^R - W_t^R \) is normally distributed with mean 0 and variance \((s-t)\), and is independent of \( F_t \). Therefore, we have

\[
A = S(t) \sum_{g=1}^{n} P_R \left[ e_{g,1} \geq Z_1, \ldots, e_{g,g-1} \geq Z_{g-1}, d_{i,g} \geq Z_{g}, e_{g,g+1} \geq Z_{g+1}, \ldots, e_{g,n} \geq Z_{n}, y_{j} \geq Z_{n+1} \right] \quad (A.8)
\]

where \( Z_i \)'s are

\[
\begin{bmatrix}
W_{i,g}^R - W_{i,1}^R \\
\sqrt{t_g - t_i} \\
\vdots \\
W_{n,g}^R - W_{n,1}^R \\
\sqrt{t_n - t_g}
\end{bmatrix}
\]

Consequently, taking \( 1 \leq i < j \leq g \) as example, we have

\[
\rho_{i,j}^{g} = \rho_{j,i}^{g} = E[Z_i Z_j] = E\left[ \frac{(W_{i,g}^R - W_{i,1}^R)}{\sqrt{t_g - t_i}} \frac{(W_{j,g}^R - W_{j,1}^R)}{\sqrt{t_g - t_j}} \right] = \frac{t_g - t_j}{t_g - t_i} \quad (A.10)
\]

We can repeat the above method to obtain covariance matrix \( \Sigma_g \) as in (22).

Therefore, the solution for \( A \) is

\[
S(t) \sum_{g=1}^{n} N_{n+1} \left[ e_{g,1}, \ldots, e_{g,g-1}, d_{i,g}, e_{g,g+1}, \ldots, e_{g,n}, y_{j}, \Sigma_g \right] = S(t) \sum_{g=1}^{n} N_{n+1} \left[ D_{i,h}^{g}, \Sigma_g \right] \quad (A.11)
\]

Similarly, (A.3) can be computed by the same technique. This completes the proof of the Theorem.
APPENDIX B

Delta and Gamma of Reset Options

To derive the Delta and Gamma of reset options, we apply the chain rule of differentiation:

\[
\frac{\partial N_{n+1}(D_{g}^{i,k}, \Sigma_{g})}{\partial S(t)} = \frac{\partial N_{n+1}(D_{g}^{i,k}, \Sigma_{g})}{\partial d_{i,g}} \frac{\partial d_{i,g}}{\partial S(t)} + \frac{\partial N_{n+1}(D_{g}^{i,k}, \Sigma_{g})}{\partial y_{k}} \frac{\partial y_{k}}{\partial S(t)}
\]  

(B.1)

where

\[
\frac{\partial d_{i,g}}{\partial S(t)} = \frac{1}{\sigma S(t)\sqrt{t_{g} - t}}, \quad \frac{\partial y_{k}}{\partial S(t)} = \frac{1}{\sigma S(t)\sqrt{T - t}}.
\]  

(B.2)

We then have the Delta and Gamma of reset options with \(m\) strike resets and \(n\) pre-decided reset dates as follows:

\[
\text{Delta}(t,S(t)) = N(y_{m}) + \sum_{i=1}^{m} \sum_{g=1}^{n} \left[ N_{n+1}(D_{g}^{i-1}, \Sigma_{g}) - N_{n+1}(D_{g}^{i}, \Sigma_{g}) \right] 
\]

\[
+ \sum_{i=1}^{m} \sum_{g=1}^{n} \left[ \frac{1}{\sigma \sqrt{t_{g} - t}} \left[ \frac{\partial N_{n+1}(D_{g}^{i-1}, \Sigma_{g})}{\partial d_{i,g}} - \frac{\partial N_{n+1}(D_{g}^{i}, \Sigma_{g})}{\partial d_{i,g}} \right] \right] 
\]

\[
+ \frac{1}{\sigma \sqrt{T - t}} \left[ \frac{\partial N_{n+1}(D_{g}^{i-1}, \Sigma_{g})}{\partial y_{i-1}} - \frac{\partial N_{n+1}(D_{g}^{i}, \Sigma_{g})}{\partial y_{i}} \right]
\]

\[
- \sum_{i=1}^{m} \sum_{g=1}^{n} K_{i}e^{-r(T-t)} \frac{1}{S(t)} \left[ \frac{1}{\sigma \sqrt{t_{g} - t}} \left[ \frac{\partial N_{n+1}(\hat{D}_{g}^{i-1}, \Sigma_{g})}{\partial \hat{d}_{i+1,g}} - \frac{\partial N_{n+1}(\hat{D}_{g}^{i-1}, \Sigma_{g})}{\partial \hat{d}_{i+1,g}} \right] \right]
\]

\[
+ \frac{1}{\sigma \sqrt{T - t}} \left[ \frac{\partial N_{n+1}(\hat{D}_{g}^{i-1}, \Sigma_{g})}{\partial \hat{y}_{i}} - \frac{\partial N_{n+1}(\hat{D}_{g}^{i-1}, \Sigma_{g})}{\partial \hat{y}_{i}} \right]
\]

\[
- \frac{K_{0}e^{-r(T-t)}}{S(t)} \left[ \sum_{g=1}^{n} \left[ \frac{1}{\sigma \sqrt{t_{g} - t}} \frac{\partial N_{n+1}(\hat{D}_{g}^{0}, \Sigma_{g})}{\partial \hat{d}_{0,g}} + \frac{1}{\sigma \sqrt{T - t}} \frac{\partial N_{n+1}(\hat{D}_{g}^{0}, \Sigma_{g})}{\partial \hat{y}_{0}} \right] \right]
\]

\[
+ \frac{K_{m}e^{-r(T-t)}}{S(t)} \left[ \sum_{g=1}^{n} \left[ \frac{1}{\sigma \sqrt{t_{g} - t}} \frac{\partial N_{n+1}(\hat{D}_{g}^{m,n}, \Sigma_{g})}{\partial \hat{d}_{m,g}} + \frac{1}{\sigma \sqrt{T - t}} \frac{\partial N_{n+1}(\hat{D}_{g}^{m,n}, \Sigma_{g})}{\partial \hat{y}_{m}} \right] \right].
\]  

(B.3)
\[ \text{Gamma}(t, S(t)) = \frac{1}{\sigma \sqrt{2\pi(T-t)}} e^{-\frac{1}{2} \frac{(t-T)^2}{\sigma^2}} + \frac{1}{\sigma S(t)} \sum_{l=1}^{m} \sum_{g=1}^{n} \left\{ \frac{1}{\sqrt{t_g - t}} \left[ \frac{\partial N_{n+1}(D_{g}^{l-1}, \Sigma_g)}{\partial d_{l,g}} - \frac{\partial N_{n+1}(D_{g}^{l}, \Sigma_g)}{\partial d_{l,g}} \right] \right\} \]

\[ + \frac{1}{\sqrt{T-t}} \left[ \frac{\partial N_{n+1}(D_{g}^{l-1}, \Sigma_g)}{\partial y_{l-1}} - \frac{\partial N_{n+1}(D_{g}^{l}, \Sigma_g)}{\partial y_{l}} \right] \}

\[ + \sum_{l=1}^{m} \sum_{g=1}^{n} [A(l, l-1, g) + B(l, l-1, g)] - \sum_{l=1}^{m} \sum_{g=1}^{n} K_e e^{-r(T-t)} \sum_{l=1}^{m} \sum_{g=1}^{n} [A(l+1, l, g) + B(l+1, l, g)] \]

\[-\frac{K_0 e^{-r(T-t)}}{\sigma^2 S(t)^2} \sum_{g=1}^{n} \left\{ \frac{1}{\sqrt{t_g - t}} \left[ \frac{\partial^2 N_{n+1}(\hat{D}_{g}^{1,0}, \Sigma_g)}{\partial d_{l,g}^2} \right] \right\} \]

\[ + \frac{1}{\sqrt{T-t}} \left[ \frac{\partial^2 N_{n+1}(\hat{D}_{g}^{1,0}, \Sigma_g)}{\partial d_{l,g} \partial y_0} - \frac{\partial^2 N_{n+1}(\hat{D}_{g}^{1,0}, \Sigma_g)}{\partial y_0 \partial y_0} \right] \}

\[ + \frac{K_m e^{-r(T-t)}}{\sigma^2 S(t)^2} \sum_{g=1}^{n} \left\{ \frac{1}{\sqrt{t_g - t}} \left[ \frac{\partial^2 N_{n+1}(\hat{D}_{g}^{0,m}, \Sigma_g)}{\partial d_{l,g}^2} \right] \right\} \]

\[ + \frac{1}{\sqrt{T-t}} \left[ \frac{\partial^2 N_{n+1}(\hat{D}_{g}^{0,m}, \Sigma_g)}{\partial d_{l,g} \partial y_m} - \frac{\partial^2 N_{n+1}(\hat{D}_{g}^{0,m}, \Sigma_g)}{\partial y_m \partial y_m} \right] \}

\[ + \sum_{l=1}^{m} \sum_{g=1}^{n} K_i e^{-r(T-t)} \left\{ \frac{1}{\sigma \sqrt{t_g - t}} \left[ \frac{\partial N_{n+1}(\hat{D}_{g}^{l+1,1}, \Sigma_g)}{\partial \hat{y}_l} - \frac{\partial N_{n+1}(\hat{D}_{g}^{l+1}, \Sigma_g)}{\partial \hat{y}_l} \right] \right\} \]

\[ \frac{1}{\sigma \sqrt{T-t}} \left[ \frac{\partial N_{n+1}(\hat{D}_{g}^{l+1,1}, \Sigma_g)}{\partial \hat{y}_l} - \frac{\partial N_{n+1}(\hat{D}_{g}^{l+1}, \Sigma_g)}{\partial \hat{y}_l} \right] \}

\[ + \frac{K_0 e^{-r(T-t)}}{S(t)^2} \left\{ \frac{1}{\sigma \sqrt{t_g - t}} \left[ \frac{\partial N_{n+1}(\hat{D}_{g}^{1,0}, \Sigma_g)}{\partial \hat{y}_0} - \frac{\partial N_{n+1}(\hat{D}_{g}^{1,0}, \Sigma_g)}{\partial \hat{y}_0} \right] \right\} \]

\[ - \frac{K_m e^{-r(T-t)}}{S(t)^2} \left\{ \frac{1}{\sigma \sqrt{t_g - t}} \left[ \frac{\partial N_{n+1}(\hat{D}_{g}^{0,m}, \Sigma_g)}{\partial \hat{y}_m} - \frac{\partial N_{n+1}(\hat{D}_{g}^{0,m}, \Sigma_g)}{\partial \hat{y}_m} \right] \right\} \] (B.4)

where

\[ A(i, j, k) = \frac{1}{\sigma \sqrt{t_k - t}} \left[ \frac{1}{\sqrt{t_k - t}} \left[ \frac{\partial^2 N_{n+1}(D_{g}^{l-1}, \Sigma_g)}{\partial d_{k,j}^2} - \frac{\partial^2 N_{n+1}(D_{g}^{l}, \Sigma_g)}{\partial d_{k,j}^2} \right] \right] \]
\[ B(i, j, k) = \frac{1}{S(t)\sigma^2\sqrt{T-t}} \left\{ \frac{1}{\sqrt{t-k}} \left[ \frac{\partial^2 N_{n+1}(D_k^{i-1}, \Sigma_k)}{\partial d_{i,k} \partial y_{i-1}} - \frac{\partial^2 N_{n+1}(D_k^{i,j}, \Sigma_k)}{\partial y_j \partial d_{i,k}} \right] + \frac{1}{\sigma \sqrt{T-t}} \left[ \frac{\partial^2 N_{n+1}(D_k^{i,j}, \Sigma_k)}{\partial y_j^2} - \frac{\partial^2 N_{n+1}(D_k^{i,j}, \Sigma_k)}{\partial d_{i,k}^2} \right] \right\}, \]  

(B.6)

and \( A(i, j, k) \) and \( B(i, j, k) \) are similar to \( A(i, j, k) \) and \( B(i, j, k) \) with the parameters \( d_{i,j} \) and \( y_j \) replaced by \( \tilde{d}_{i,j} \) and \( \tilde{y}_j \), respectively.

By observing (B.3) and (B.4), we see the key elements for computing the hedge ratio are \( \partial N_{n+1}(D^{i,h}_g, \Sigma_g) / \partial d_{i,g} \), \( \partial^2 N_{n+1}(D^{i,h}_g, \Sigma_g) / \partial d_{i,g} \partial y_h \) and \( \partial^2 N_{n+1}(D^{i,h}_g, \Sigma_g) / \partial d_{i,g}^2 \). To derive the derivatives, as Curnow and Dunnett (1961) pointed out\(^6\), we have

\[ N_{n+1}(D^{i,h}_g, \Sigma_g) = \int_{-\infty}^{d_{i,g}} N_x \left[ \frac{\frac{e_{i,j} - \rho_{jg}^g d_{i,g}}{\sqrt{1-(\rho_{jg}^g)^2}}, \frac{y_h - \rho_{ng}^g d_{i,g}}{\sqrt{1-(\rho_{ng}^g)^2}}, \left\{ \rho_{qk}^g \right\}_{n+1} }{f(d_{i,g})} \right] f(x) dx \]  

(B.7)

where

\[ \rho_{qk}^g = \frac{\rho_{qg}^g - \rho_{gk}^g \rho_{kg}^g}{\sqrt{1-(\rho_{qg}^g)^2} \sqrt{1-(\rho_{kg}^g)^2}} \]  

(B.8)

and \( f(\cdot) \) is the standard normal probability density function. Hence,

\[ \frac{\partial N_{n+1}(D^{i,h}_g, \Sigma_g)}{\partial d_{i,g}} \]  

can be calculated as follows:

\[ \frac{\partial N_{n+1}(D^{i,h}_g, \Sigma_g)}{\partial d_{i,g}} = N_x \left[ \frac{e_{i,j} - \rho_{jg}^g d_{i,g}}{\sqrt{1-(\rho_{jg}^g)^2}}, \frac{y_h - \rho_{ng}^g d_{i,g}}{\sqrt{1-(\rho_{ng}^g)^2}}, \left\{ \rho_{qk}^g \right\}_{n+1} \right] f(d_{i,g}) \]  

(B.9)

Then, following the similar procedure, we can straightforwardly obtain the derivatives

\[ \frac{\partial^2 N_{n+1}(D^{i,h}_g, \Sigma_g)}{\partial d_{i,g} \partial y_h} \]  

and \( \frac{\partial^2 N_{n+1}(D^{i,h}_g, \Sigma_g)}{\partial d_{i,g}^2} \).

\(^6\) The similar technique is also used to study the hedge ratio of discrete barrier options by Wei (1998).
REFERENCES


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