Chapter 5 & 6
The Time Value of Money

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Topics Covered

- Future Values
- Present Values
- Multiple Cash Flows
- Perpetuities and Annuities
- Effective Annual Interest Rate
- Loan types and amortization
- Applications
Key Concepts and Skills

- What is present value? What is future value?
- Compute the present value and future value of multiple cash flows
- Compute loan payments and interest rate
- Understand how interest rates are quoted
- Use a financial calculator to solve the time value of money problems

Future Values

- Future Value (FV)
  - Amount to which an investment will grow after earning interest
- Simple Interest
  - Only the original investment earns interests
- Compound Interest
  - Both original investment and its interest earn interests
Simple Vs. Compound Interest

- What is the total money of investing $100 at 10% annual rate for 3 years at simple and compound interest
- Simple interest:
  - Total value: $10 + $10 + $10 + $100 = $130
- Compound interest:
  - Total value: $100 \times (1 + 10\%)^3 = 133.1

\[ FV = \$100 \times (1 + r)^n \]

Compound Interest

- In simple interest, int. in each year = 100 * 10\% = 10
- In compounding, interests can also create interests
- Int. in yr 1 = 100 * 10\% = 10
- Int. in yr 2 = 100 * 10\% + 10 * 10\% = 11
- Int. in yr 3 = 100 * 10\% + 10 * 10\% + 11 * 10\% = 12.1
Future Values with Compounding

Manhattan Island Sale

- Peter Minuit bought Manhattan Island for $24 in 1626. Was this a good deal?
- To answer, determine $24 is worth in the year 2020, compounded at 8%.
- \[ FV = 24 \times (1 + 0.08)^{394} = 354.1 \text{ trillion} \]

FYI - The value of Manhattan Island is well below this figure.
Present Values

- Present value (PV)
  - Value today of a future cash flow
- Discount factor
  - Present value of a $1 future payment
- Discount rate
  - Interest rate used to compute present values of future cash flows

PV and FV

- $FV = PV (1+r)^n$
- $PV = FV / (1+r)^n$
- Discount Factor ($DF$) = $PV$ of $1 = 1 / (1+r)^n$
- $PV = FV * DF$

- Increasing interest rate and time increases FV
- Increasing interest rate and time decreases PV
Paying for Education

- Just had a baby. You plan that the baby will get a Bachelor’s degree, and will go to a top-tier 2-year MBA program when baby is 24.
- The estimated future cost of the MBA at $60,000 for year 1 and $70,000 for year 2.
- Today, you want to finance both years of baby’s MBA program with one payment (deposit) into an account paying 6% interest compounded annually.
- How large must this deposit be?

Paying for Education

- Cash outflow
  - T=24, FV = 60,000
  - T=25, FV = 70,000
- Annual rate = 6%
- Present value of two future cash flows
  \[ = \frac{60,000}{(1+6\%)^{24}} + \frac{70,000}{(1+6\%)^{25}} = 31,128 \]
- We can verify this number by:
  \[ [31,128 \times (1.06)^{24} - 60,000] \times (1.06) = 70,000 \]
PV of Multiple Cash Flows

- PVs can be added together to evaluate multiple cash flows.

\[ PV = \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \ldots \]

PV of Multiple Cash Flows

- Your auto dealer gives you two choices, if your cost of money is 8%, which do you prefer?
  - pay $15,500 cash now, or
  - make three payments:
    - $8,000 now
    - $4,000 at the end of the following two years

Immediate payment 8,000

\[ PV_1 = \frac{4000}{(1+.08)^1} = 3,703 \]

\[ PV_2 = \frac{4000}{(1+.08)^2} = 3,429 \]

Total \( PV \) = $15,133
Annuities

- Annuities
  - A series of equal periodical cash flows
- Ordinary Annuity
  - Each cash flow is at the end of each period
- Annuity Due
  - Each cash flow is at the beginning of each period

PV and FV of Annuities

- Annuity factor
  \[ AF = \frac{1}{r} - \frac{1}{r(1+r)^n} \]
  - Present value of a \( n \)-year annuity of $1
  - For example, 3-year annuity of $1 @10\% = \\frac{1}{(1+0.1)^1} + \frac{1}{(1+0.1)^2} + \frac{1}{(1+0.1)^3} = \frac{1}{0.1} - \frac{1}{0.1(1+0.1)^3} = 2.4869
- PV of ordinary annuity = \( C \* AF \)
- PV of annuity due = \( C \* AF \* (1+r) \)
- FV of ordinary annuity = \( C \* AF \* (1+r)^n \)
- FV of annuity due = \( C \* AF \* (1+r)^n \* (1+r) \)
Perpetuities

- Perpetuity
  - A stream of cash payments that never ends
  - Annuity that goes on forever

- PV of annuity
  - \( PV = C \times AF = C \left( \frac{1}{r} - \frac{1}{r(1 + r)^n} \right) \)
  - when \( n \rightarrow \infty, \frac{1}{r(1 + r)^n} \rightarrow 0 \)

- PV of Perpetuity = \( \frac{C}{r} \)

How easy is retirement?

- Mr. Buffett, currently age 75, wants to retire at age 90. Once he retires, he wants to withdraw $300 million at the beginning of each year for 10 years from a special offshore account that will pay 20% annually.

- In order to fund his retirement, Mr. Buffett will make 15 equal end-of-the-year deposits in this same special account that will pay 20% annually.

- How large of an annual deposit must be made to fund Mr. Buffett’ retirement plans?
Easy retirement

- First calculate the PV of the 10 Yr payments after retirement
  - $PMT = 300$ mil at beginning, $r = 20\%$, $N = 10$
  - $PV = 1509.3$ mil
- Second, calculate the constant payment to meet the fund requirement
  - $FV = 1509$ mil, $r = 20\%$, $N = 15$
  - $PMT = 20.95$ mil at end of each period

PV of Perpetuity

- You want to create an endowment to fund a football scholarship, which pays $15,000 per year, forever, how much money must be set aside today if the rate of interest is 5%?
  - $PV = 15,000 / 0.05 = 300,000$
Effective Interest Rates

- Annual Percentage Rate (APR)
  - Annualized interest rate based on simple interest
- Effective Annual Rate (EAR)
  - Annualized interest rate based on compound interest
  - Actual rate interest earned/paid
  - $APR = \text{periodic rate} \times m$
  - $EAR = (1+\frac{APR}{m})^m - 1$
    - $m$ is number of compounding periods per year

Credit Card EAR

- A credit card charges 18% APR compounded monthly
- What is the EAR?

- $APR = 18\%$, and $m=12$, so periodic rate = 1.5%
- $EAR = (1+1.5\%)^{12} - 1 = 19.56\%$
You want to buy a Lexus minivan which costs $33,500. You will finance $33,500.

Lexus offers two choices:
- $1,500 rebate, plus 0% APR financing for 48 months, or
- $4,500 rebate, plus 4.29% APR financing for 48 months

Which one would give you the lowest monthly payment?

Option 1
- Loan amount = $33,500 - $1,500 = $32,000
- APR = 0%, so $r = 0$, $N = 48$, $PV = 32,000$
- PMT = $32,000 / 48 = $666.67$

Option 2
- Loan amount = $33,500 - $4,500 = $29,000
- $r = 4.29% / 12 = 0.3575\%$, $N = 48$, $PV = 29,000$
- PMT = $658.56$

So, option 2 is cheaper.
Loan Types

- Pure discount loan
  - Pays the lump sum on expiration, no periodical payment
  - Example: T-bills

- Interest-only loan
  - Pays interests in each period and principal on expiration
  - Examples: corporate bonds, T-bonds

- Amortized loan
  - Amortizing the principal into periodical payments
  - Example: mortgage

Pure Discount Loans

- If a T-bill promises to repay $10,000 in 12 months and the market interest rate is 7 percent, how much should the bill sell for in the market now?

- \( PV = \frac{10,000}{1.07} = 9345.79 \)
Interest-Only Loan

- Consider a 5-year, interest only loan with a 7% interest rate. The principal amount is $10,000. Interest is paid annually.

- What would the stream of cash flows be?
  - Years 1 – 4: Interest = 0.07(10,000) = 700
  - Year 5: Interest + principal = 10,700

Amortized Loan with Fixed Principal Payments

- Amortizing the principal into the same periodical payments
- The total payment in each period is reducing over time
- Consider a $50,000, 10 year loan at 8% interest. The loan agreement requires the firm to pay $5,000 in principal each year plus interest for that year.
Amortized Loan with Fixed Principal Payments

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<tr>
<th>Year</th>
<th>Beginning Balance</th>
<th>Interest Payment</th>
<th>Principal Payment</th>
<th>Total Payment</th>
<th>Ending Balance</th>
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Amortized Loan with Fixed Payment

- The total payment is the same in each period
- Consider a 4 year loan with annual payments. The interest rate is 8% and the principal amount is $5000.
- What is the annual payment?
  - N = 4, r = 8, PV = 5000
  - PMT = -1509.60
## Amortized Loan with Fixed Payment

<table>
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<tr>
<th>Year</th>
<th>Beginning Balance</th>
<th>Total Payment</th>
<th>Interest Paid</th>
<th>Principal Paid</th>
<th>Ending Balance</th>
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</thead>
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<td>400.00</td>
<td>1,109.60</td>
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Note: The ending balance of .01 is due to rounding. The last payment would actually be 1,509.61.