Linear and Affine Functions

Use the notation \( f : \mathbb{R}^m \to \mathbb{R}^n \) to indicate a function whose domain is a subset of \( \mathbb{R}^m \) and whose range is a subset of \( \mathbb{R}^n \). In other words, \( f \) takes a vector with \( m \) coordinates for input and returns a vector with \( n \) coordinates. For example, the function:

\[
 f(x, y, z) = (\sin(x + y), 2x^2 + z)
\]

is a function from \( \mathbb{R}^3 \to \mathbb{R}^2 \).

**Definition 1** We say a function \( L : \mathbb{R}^m \to \mathbb{R}^n \) is linear if (1) for any vectors \( x \) and \( y \) in \( \mathbb{R}^m \), \( L(x + y) = L(x) + L(y) \), and (2) for any vector \( x \) in \( \mathbb{R}^m \) and scalar \( a \), \( L(ax) = aL(x) \).

**Example 2** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is defined by \( f(x) = 3x \). Then for any \( x \) and \( y \) in \( \mathbb{R} \),

\[
 f(x + y) = 3(x + y) = 3x + 3y = f(x) + f(y),
\]

and for any scalar \( a \),

\[
 f(ax) = 3ax = af(x).
\]

Thus \( f \) is linear.

**Example 3** Suppose \( L : \mathbb{R}^2 \to \mathbb{R}^3 \) is defined by \( L(x_1, x_2) = (2x_1 + 3x_2, x_1 - x_2, 4x_2) \). Then if \( \mathbf{x} = (x_1, x_2) \) and \( \mathbf{y} = (y_1, y_2) \) are vectors in \( \mathbb{R}^2 \),

\[
 L(\mathbf{x} + \mathbf{y}) = L(x_1 + y_1, x_2 + y_2)
 = (2(x_1 + y_1) + 3(x_2 + y_2), x_1 + y_1 - (x_2 + y_2), 4(x_2 + y_2))
 = (2x_1 + 3x_2, x_1 - x_2, 4x_2) + (2y_1 + 3y_2, y_1 - y_2, 4y_2)
 = L(x_1, x_2) + L(y_1, y_2)
 = L(\mathbf{x}) + L(\mathbf{y}).
\]

Also, for \( \mathbf{x} = (x_1, x_2) \) and any scalar \( a \), we have:

\[
 L(ax) = L(ax_1, ax_2)
 = (2ax_1 + 3ax_2, ax_1 - ax_2, 4ax_2)
 = a(2x_1 + 3x_2, x_1 - x_2, 4x_2)
 = aL(\mathbf{x}).
\]
Thus $L$ is linear.

**Definition 4** We say a function $A : \mathbb{R}^m \to \mathbb{R}^n$ is affine if there is a linear function $L : \mathbb{R}^m \to \mathbb{R}^n$ and a vector $b$ in $\mathbb{R}^n$ such that $A(x) = L(x) + b$ for all $x$ in $\mathbb{R}^m$.

In other words, an affine function is just a linear function plus a translation. From our knowledge of linear functions, it follows that if $A : \mathbb{R}^m \to \mathbb{R}^n$ is affine, then there is an $n \times m$ matrix $M$ and a vector $b$ in $\mathbb{R}^n$ such that:

$$A(x) = Mx + b,$$

for all $x$ in $\mathbb{R}^m$. In particular, if $f : \mathbb{R} \to \mathbb{R}$ is affine, then there are real numbers $m$ and $b$ such that:

$$f(x) = mx + b,$$

for all real numbers $x$. 